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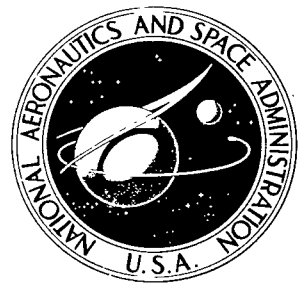

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MOTION OF AN ARTIFICIAL SATELLITE IN AN ECCENTRIC GRAVITATION FIELD

by V. G. Demin

Nauka Press, Moscow, 1968

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By V. G. Demin

Translation of "Dvizheniye Iskusstvennogo Sputnika v Netsentral'nom Pole Tyagoteniya"
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MOTION OF AN ARTIFICIAL SATELLITE IN AN ECCENTRIC GRAVITATION FIELD

V. G. DEMIN

ABSTRACT. In this book are presented analytical and qualitative methods for studying the motion of artificial satellites in a central gravitational field, and also methods for constructing the intermediate orbits of a satellite of an axisymmetrical planet. Basic attention is directed to the problem of two immobile centers and to its modifications which may be relevant to the science of celestial ballistics. We present a classification of the forms of motion encountered in this problem, and make a detailed study of the most important class of satellite trajectories. Working formulas suitable for the needs of long-range prediction of the motion of artificial satellites are given. A partial case of the three-body problem, and also certain model problems of celestial ballistics, are considered.

The qualitative properties of motion (stability, periodicity and near-periodicity) are approached mainly by the classical methods of H. Poincare and A. M. Lyapunov, but also with the help of certain results obtained by V. I. Arnol'd. The details of the applications of these methods are illustrated on the basis of a number of specific examples selected from the dynamics of space flight. The derivation of the equations of motion, along with their transformation and solution are also arrived at on the basis of the methods of analytical dynamics.

The book includes the necessary information from the theory of the Newtonian potential, analytical dynamics, and the qualitative methods of celestial mechanics. The book is intended for scientists, engineers, and students concerned with celestial ballistics and with celestial mechanics. There are four tables, 21 illustrations, and a bibliography of 209 items.

Introduction /7*

This book is devoted to problems which lie on the boundary between two sciences.-- classical celestial mechanics, and celestial ballistics, which is still in its infancy. Although formed at the very heart of celestial mechanics, the new science of celestial ballistics, owing to the peculiar nature of the problems facing it, makes wide use of the methods employed in various other branches of mechanics, in addition to those of the parent science. The tasks of celestial ballistics have been strongly conditioned by the adoption of the methods used in the theory of optimal processes and also those of the theory of automatic control. In addition, the new methods of computer mathematics, based as they are on the extensive possibilities inherent in high-speed electronic computers, have had a very powerful influence on the young science of celestial ballistics.

The use of electric computers, it is true, has played a vital role in the successful lanuching of spaceships, as well as in adjusting their orbits. It would, however, be erroneous to assume that the development of numerical methods alone is all that is needed for the long-range advancement of celestial ballistics. At the first stage of development of celestial ballistics, it was necessary to conduct the "planning" of orbits through analysis of hundreds of trajectories obtained by numerical integration on electronic computers. It was therefore entirely natural to untake the task of developing a sufficiently simple approximation theory which would be at once convenient and economical. Such a theory would make it possible without resorting to cumbersome calculations, to arrive at the initial conditions for the orbits of spaceships with pre-assigned properties. The formulation of a precise analytical theory would make it possible, without recourse to /8 numerical integration, to achieve a long-range prediction of satellite motion. Finally, such a theory is necessary in determining the parameters of the gravitational fields of the earth, other planets, and the moon.

So-called qualitative methods are able to supply a good deal of information regarding the dynamics of spacecraft. One may mention, in this connection, the importance of these methods in determining the initial conditions for periodic or near-periodic orbits, and for other satellite motions which are stable in one sense or another. Finally, purely qualitative study of the structure of the total integral of a non-integrable problem makes it possible to select the most suitable method of numerical solution. Therefore, the most expedient objective is a rational combination of numerical, analytical and claritative methods.

On the basis of these various considerations, the author of the present volume has included a number of "model" problems of celestial mechanics which in first approximation offer the possibility of a simple means of performing the necessary calculations, and which serve as a basis for the construction of precise theories of perturbed motion. The solution of these particular problems has served as an education for both Soviet and foreign scientists. It is of this that the accumulation of so-called model problems represents

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* Numbers in the margin indicate pagination in the foreign text.

one aspect of the development of celestial ballistics. In the language of celestial mechanics, such problems are referred to as "simplified", or "unperturbed", and the corresponding orbits are referred to as "intermediate". We can justifiably say, therefore, that the main subject of the present volume is the theory of the intermediate orbits of spaceships.

From among the many problems of celestial ballistics we select here those which have to do with the study of a spaceship conceived as a material point moving within a central gravitational field. It is natural that we should include in this group of problems not only the motion of an artificial satellite within the gravitational field of a spheroidal planet, but also the partial case of the circular three-body problem, the problem of the motion of a spaceship in a Newtonian central field in the presence of light pressure, and a number of others. Our tasks can rightly be called a limited problem of celestial mechanics. Mathematically, it can be formulated in the following system of differential equations:

$$\begin{aligned} \ddot{x} - 2n\dot{y} - n^2 x &= U'_x, \\ \ddot{y} + 2n\dot{x} - n^2 y &= U'_y, \\ \ddot{z} &= U'_z, \end{aligned}$$

in which /9

$$U = \frac{fm}{r} + \mu R(x, y, z, t, \mu),$$

where f is the gravitational constant, m is the mass of the gravitating body, r is the radius vector of the moving point, n is the angular rotational velocity of the coordinate system, μR is the perturbation function, and μ is a minor parameter.

Our principal attention will be directed toward the motion of an artificial satellite within the gravitational field of a spheroidal planet. Actually, the amount of research done on this particular problem is so great that we cannot attempt, in a work of this limited scope, to present even an outline of the main efforts which have been made. We therefore limit ourselves to a particular problem, namely the generalized problem of two immobile centers. It is this problem, which has attracted the attention of so many specialists, that I regard as most rewarding in the present line of endeavor. One may expect that it will assume in celestial mechanics a position analogous to that of the two-body problem, the circular three-body problem, etc. Apart from this, we give here certain alternative variants for the formulation of a theory of satellite motion. The author has attempted, so far as possible, to present a systematic and complete statement of the results thus far obtained in this field, in order to fill in the gaps existing in the published literature which stand in the way of implementing a new theory.

The dimensions of the present volume, unfortunately, have not enabled me

to include all of the available theoretical information, let alone all of the practical material, relating to the Delavnay method. Being a technique common to the theory of perturbations, the Delavnay method leads to a formal solution in purely trigonometric form: in other words, at least on the practical plane, it enables us to overcome the difficulty which is basic in all celestial mechanics problems -- namely, the appearance of secular terms in an approximate solution, which are the scourge of any formal-analytical theory. In celestial ballistics the Delavnay method has been successfully applied to the problem of the motion of artificial satellites by D. Brower and Y. Kozai. /10

The present author attributes great significance to the development of qualitative methods in the analysis of the equations of celestial ballistics, and in this connection has given here a brief description of two fundamental methods -- namely, the method of the minor parameter of Poincaré, and the second method of A. M. Lyapunov. The manner in which these methods are applied in celestial ballistics is illustrated in a number of individual problems. It has also been found useful to include discussions of the stability and conditional periodicity of artificial satellites, on the basis of results obtained by V. I. Arnol'd in his study of Hamiltonian systems.

The presentation of the theory of intermediate orbits, like that of a number of the problems of qualitative analysis, is tied in with the use of the apparatus of analytical mechanics. The author has therefore found it useful to include in the present volume a special chapter on the subject of analytical mechanics.

A good portion of the book is drawn from lectures given by the author in previous years at the physical and mechanical-mathematical faculties of the Moscow University, and also at the Faculty of Physico-Mathematical and Natural Sciences of the P. Lumumba University of International Friendship.

The author feels obligated to express profound gratitude to the members of the Department of Celestial Mechanics and Gravimetry of the Moscow State University, and more particularly to Professor G. N. Duboshin and Professor B. M. Shchigolev, for their helpful discussion on a number of the subjects covered in this book. The author recalls with great satisfaction his collaboration with Ye. A. Grebenikov and Ye. P. Aksenov on problems dealt with here; he also expresses his sincere gratitude to D. Ye. Okhotsimskiy, corresponding member of the USSR Academy of Sciences, and to Professor V. V. Beletskiy for their concern and kindly advice, to A. L. Kunitsyn for his extensive help and useful criticism and to G. T. Arazov for his assistance in compiling the manuscript.

The author will be grateful for any additional remarks and advice.

CHAPTER I

NECESSARY INFORMATION FROM ANALYTICAL DYNAMICS

§ 1. Lagrangian Equations

/11

Let us consider the motion of a mechanical system consisting of s material points. We shall assume that on this system there have been imposed p holonomic, ideal, bilaterial links. We shall use the symbols $m_1, m_2, \ldots, m_s$ to denote the masses of the material points, and the symbols $r_1, r_2, \ldots, r_s$ to denote their radius-vectors within a certain inertial system of Cartesian coordinates. Finally, we shall use the symbol F_ν to denote the resultant of all the active forces applied to the ν-th material point.

In the case of natural dynamic systems, applied forces are described with the help of a special function-potential, or force function. This function is independent of time and of the coordinates of the points of the system. A. Mayer [1-2] extended the concept of the potential to a more general case in which the forces F_ν in a rectangular system of Cartesian coordinates x, y, z are defined by the following relationships

$$\mathbf{F}_\nu = \operatorname{grad}_{(x_\nu, y_\nu, z_\nu)} U - \frac{d}{dt}\left[\operatorname{grad}_{(\dot{x}_\nu, \dot{y}_\nu, \dot{z}_\nu)} U\right], \tag{1.1}$$

where the function U depends not only upon time and upon the coordinates of the material points but also upon the velocity components. It has been agreed to call this function U as the Mayer potential[1].

This extention of the concept of the potential has been found convenient in describing motion in the presence of gyroscopic forces. The Mayer potential in particular is adaptable in the study of motion within noninertial systems in which a Coriolis force of inertial arise. In celestial ballistics the Mayer potential can be applied in the limited circular three-boiled problem (for example, in the problem of reaching the Moon), in the problem of /12 the motion of a satellite in the gravitational field of a rotating planet, etc.

Let us write the equations of motion, assuming, also, that in addition to the forces described by equation (1.1), other forces Φ_ν without a potential are applied to the points of the system:

[1] The general conditions for the existence of a potential are given in the writings of Helmholtz [3] and Hirsch [4].

$$m_\nu \ddot{\mathbf{r}}_\nu = \mathbf{F}_\nu + \mathbf{\Phi}_\nu + \mathbf{R}_\nu \quad (\nu = 1, 2, \ldots, s), \tag{1.2}$$

where R_ν is the resultant of all passive forces which act upon the ν-th point.

Thus since we have assumed that the links are ideal, then for any possible displacement the work of the forces of reaction are equal to zero:

$$\sum_{\nu=1}^{s} \mathbf{R}_\nu \cdot \delta \mathbf{r}_\nu = 0. \tag{1.3}$$

Let us transform equations (1.2) to the generalized (Lagrangian) coordinates $q_1, q_2, \ldots, q_n$ ($n = 3s - p$) defined by the following relationship

$$\mathbf{r}_\nu = \mathbf{r}_\nu (t, q_1, q_2, \ldots, q_n). \tag{1.4}$$

If within formula (1.3), in place of R_ν we substitute their expressions from equation (1.2), then we arrive at this relationship:

$$\sum_{\nu=1}^{s} (m_\nu \ddot{\mathbf{r}}_\nu - \mathbf{F}_\nu - \mathbf{\Phi}_\nu) \cdot \delta \mathbf{r}_\nu = 0. \tag{1.5}$$

We then calculate the variations in radius-vectors r_ν with the help of the formulas of transformation (1.4):

$$\delta \mathbf{r}_\nu = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_\nu}{\partial q_i} \delta q_i. \tag{1.6}$$

Substituting in equation (1.5) the values obtained for variations, and changing the order of summation, we arrive at

$$\sum_{i=1}^{n} \left\{ \sum_{\nu=1}^{s} (m_\nu \ddot{\mathbf{r}}_\nu - \mathbf{F}_\nu - \mathbf{\Phi}_\nu) \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_i} \right\} \delta q_i = 0. \tag{1.7}$$

From equation (1.7), by reason of the independence of the variations of the generalized coordinates, we can state that /13

$$\sum_{\nu=1}^{s} \left(m_\nu \ddot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_i} - \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_i} - \mathbf{\Phi}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_i} \right) = 0, \tag{1.8}$$

where $i = 1, 2, \ldots, n$.

Making use of the symbols used in equation (1.1), we can rewrite

equation (1.8) in the following form:

$$\sum_{\nu=1}^{s}\left\{m_\nu\ddot{\mathbf{r}}_\nu\cdot\frac{\partial\mathbf{r}_\nu}{\partial q_i}+\frac{d}{dt}\left[\mathrm{grad}_{(\dot{x}_\nu,\,\dot{y}_\nu,\,\dot{z}_\nu)}U\right]\cdot\frac{\partial\mathbf{r}_\nu}{\partial q_i}-\right.$$
$$\left.-\mathrm{grad}_{(x_\nu,\,y_\nu,\,z_\nu)}U\cdot\frac{\partial\mathbf{r}_\nu}{\partial q_i}-\Phi_\nu\cdot\frac{\partial\mathbf{r}_\nu}{\partial q_i}\right\}=0\ \ (i=1,2,\ldots,n). \tag{1.9}$$

Upon differentiating the transformation formulas (1.4), it is not difficult to establish that the following identities will hold

$$\frac{\partial\mathbf{r}_\nu}{\partial q_i}=\frac{\partial\dot{\mathbf{r}}_\nu}{\partial\dot{q}_i}\quad(i=1,2,\ldots,n;\ \nu=1,2,\ldots,s).$$

Then, with the help of these identities the expression $\sum m_\nu\ddot{\mathbf{r}}_\nu\cdot\frac{\partial\mathbf{r}_\nu}{\partial q_i}$ can be transformed as follows:

$$\sum_{\nu=1}^{s}m_\nu\ddot{\mathbf{r}}_\nu\cdot\frac{\partial\mathbf{r}_\nu}{\partial q_i}=\sum_{\nu=1}^{s}\left[\frac{d}{dt}\left(m_\nu\dot{\mathbf{r}}_\nu\cdot\frac{\partial\mathbf{r}_\nu}{\partial q_i}-m_\nu\dot{\mathbf{r}}_\nu\cdot\frac{\partial\dot{\mathbf{r}}_\nu}{\partial q_i}\right)\right]=$$
$$=\frac{d}{dt}\left[\sum_{\nu=1}^{s}m_\nu\dot{\mathbf{r}}_\nu\cdot\frac{\partial\dot{\mathbf{r}}_\nu}{\partial\dot{q}_i}\right]-\frac{1}{2}\sum_{\nu=1}^{s}\frac{\partial}{\partial q_i}(m_\nu\dot{\mathbf{r}}_\nu^2)=$$
$$=\frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}_i}\left(\sum_{\nu=1}^{s}\frac{m_\nu\dot{\mathbf{r}}_\nu^2}{2}\right)\right]-\frac{\partial}{\partial q_i}\left(\sum_{\nu=1}^{s}\frac{m_\nu\dot{\mathbf{r}}_\nu^2}{2}\right). \tag{1.10}$$

Denoting with the symbol T the kinetic energy of the system

$$T=\frac{1}{2}\sum_{\nu=1}^{s}m_\nu\dot{\mathbf{r}}_\nu^2, \tag{1.11}$$

in place of equation (1.10) we obtain

$$\sum_{\nu=1}^{s}m_\nu\ddot{\mathbf{r}}_\nu\cdot\frac{\partial\mathbf{r}_\nu}{\partial q_i}=\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right)-\frac{\partial T}{\partial q_i}. \tag{1.12}$$

In analogous fashion we transform the second and the third terms of the left-hand member of equation (1.9) /14

$$\sum_{\nu=1}^{s}\left\{\frac{\partial\mathbf{r}_\nu}{\partial q_i}\cdot\frac{d}{dt}\left[\mathrm{grad}_{(\dot{x}_\nu,\,\dot{y}_\nu,\,\dot{z}_\nu)}U\right]-\frac{\partial\mathbf{r}_\nu}{\partial q_i}\cdot\mathrm{grad}_{(x_\nu,\,y_\nu,\,z_\nu)}U\right\}=\frac{d}{dt}\left(\frac{\partial U}{\partial\dot{q}_i}\right)-\frac{\partial U}{\partial q_i}. \tag{1.13}$$

Finally, we introduce the quantities

$$Q_i = \sum_{\nu=1}^{s} \Phi_\nu \cdot \frac{\partial r_\nu}{\partial q_i}, \tag{1.14}$$

which in the science of mechanics are referred to as generalized forces.

If in equation (1.9) we make the appropriate substitutions from equations (1.12) through (1.14), we arrive at the following:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} + \frac{\partial U}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - \frac{\partial U}{\partial q_i} = Q_i \quad (i = 1, 2, \ldots, n). \tag{1.15}$$

Introducing the Lagrangian function

$$L = T + U, \tag{1.16}$$

we finally arrive at the following system of differential equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad (i = 1, 2, \ldots, n), \tag{1.17}$$

which are referred to as Lagrangian equations of the second type.

If the motion takes place only under the influence of potential forces, that is, if $Q_i \equiv 0$ (i = 1, 2,..., n), then equation (1.17) will assume a particularly simple form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, 2, \ldots, n). \tag{1.18}$$

We should note that the kinetic energy T which enters into the Lagrangian function will assume the following structure in the general case:

$$T = T_2 + T_1 + T_0, \tag{1.19}$$

where T_2, T_1, T_0 are the homogeneous forms with respect to the generalized velocities $\dot{q}_i$ of the second, first and zero degrees, respectively[1]. /15

[1] Textbooks on mechanics (for example, [5]), erroneously maintain that for stationary links, $T_1 = T_0 = 0$. As a matter of fact, even with stationary links the kinetic energy of the system T may contain the terms T_1 and T_0, if the formulas of transformation to Lagrangian coordinates (1.4) reflect the factor of time in an explicit manner.

Example: Let us consider the motion of a passively gravitating point (that is, attracted but not attracting) of mass m within the Newtonian gravitational field of an absolute solid body which is rotating with a constant angular velocity n around a permanent axis. We shall assume that the rectangular Cartesian coordinate system Oxyz is rigidly affixed to the solid body. The origin of coordinates coincides with the center of mass O of the body; the equatorial plane of the body is taken as the basic coordinate plane, and the z-axis of the system lies along the body's axis of rotation, in the direction of the north pole (Figure 1)

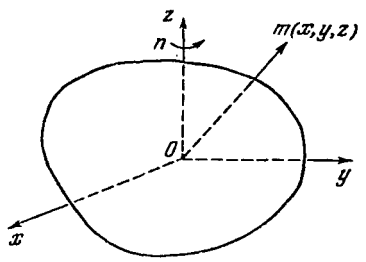

Figure 1.

The gravitational potential of the body with respect to the moving point will be designated by U(x, y, z). Within the chosen coordinate system, the moving point is under the influence not only of Newtonian gravitation, but also Coriolis and centrifugal forces of inertia. The Coriolis force $\{2n\dot{y};\ -2n\dot{x},\ 0\}$ can be defined by the Mayer potential U_1:

$$U_1 = n(x\dot{y} - y\dot{x}), \tag{1.20}$$

and the potential of inertial centrifugal force $(n^2x;\ n^2y;\ 0)$ is defined by the formula

$$U_2 = \frac{n^2}{2}(x^2 + y^2). \tag{1.21}$$

The kinetic energy T, with respect to the mass of the point in its motion within the chosen coordinate system, is

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2). \tag{1.22}$$

Then the Lagrangian of the problem, in correspondence with (1.1) and (1.16), is written in this form:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + n(x\dot{y} - y\dot{x}) + \frac{n^2}{2}(x^2 + y^2) + U(x, y, z). \tag{1.23}$$

From (1.18) and (1.23) we obtain the equations of motion of the problem: /16

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} - n^2x &= \frac{\partial U}{\partial x}, \\ \ddot{y} + 2n\dot{x} - n^2y &= \frac{\partial U}{\partial y}, \\ \ddot{z} &= \frac{\partial U}{\partial z}. \end{aligned}\right\} \tag{1.24}$$

In the restricted circular three-body problem, which formed the basis of the study of the dynamics of flight to the Moon [6, 7], the equations of motion assume an analogous form, as they also do in the problem of the motion of distant artificial earth satellites, with allowance for lunar-solar perturbations.

This classical problem consists in a study of the motion of a passively gravitating point which is attracted to two other material points on the basis of Newton's law of gravitation; here we shall consider those points to be A (the Earth) and B (the Moon). It will be assumed, further, that the Earth and the Moon are rotating around a common center of mass in Keplerian orbits at a distance a from each other, the center of mass possessing a mean motion n.

We shall consider a uniformly rotating rectangular coordinate system whose basic plane coincides with the Earth-Moon orbital plane. The origin of coordinates will be placed at the center of inertia of the Earth-Moon system, and the x-axis will pass through A and B (Figure 2). The equations of motion will be as in (1.24). The gravitational potential U is then defined by the formula

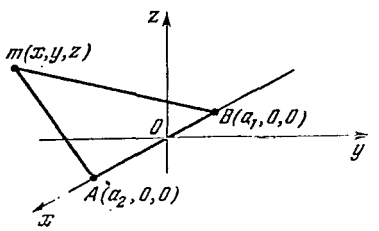

Figure 2.

$$U=\frac{fm_1}{\sqrt{(x-a_1)^2+y^2+z^2}}+\frac{fm_2}{\sqrt{(x-a_2)^2+y^2+z^2}}, \qquad (1.25)$$

in which f is the gravitational constant, and m_2 and m_1 are the masses of the moon and the earth, respectively. Since the origin of coordinates is placed at the center of inertia of the Earth-Moon system, then the following equalities are justified for a_1 and a_2:

$$\left.\begin{aligned} a_1&=\frac{m_2 a}{m_1+m_2},\\ a_2&=-\frac{m_1 a}{m_1+m_2}. \end{aligned}\right\} \qquad (1.26)$$

§ 2. Characteristic Functions in Dynamics /17

From equations (1.18) it is evident that in order to describe the motion of a mechanical system in the case of potential forces, one need only construct, in a particular manner, a certain function of generalized coordinates and velocities -- namely the Lagrangian. Functions of this type, which may be used for the description of motion, we shall refer to as "characteristic functions"[1]. Another so-called characteristic function widely used in

[1] The term "characteristic motion" is used here in a broader sense than the one generally accepted (See for example, [8]).

analytical dynamics is the Hamiltonian function, or Hamiltonian. Less well-known characteristic functions are those of V. M. Tayevskiy [8, 9] and K. M. Raitzin [10, 11]. Conversion from one of these functions to another is achieved through transformation of the generalized coordinates or the generalized velocities in new variables; here, all n generalized coordinates (or velocities) are subjected to transformations. One natural exception to this is the Routh transformation [12], which leads to a mixed Lagrangian-Hamiltonian form of the equations of motion. If the generalized coordinates and velocities are broken down into several groups, and each group is subjected to transformation, we arrive at the most general form of the characteristic function, and thereby at the most general form of the equations of motion. This form has been pointed out in an article [13].

Let the motion of the mechanical system be defined by equations (1.18). We can then transform the generalized coordinates $q_{l+1}, q_{l+2}, \ldots, q_n$ and generalized velocities $\dot{q}_{k+1}, \dot{q}_{k+2}, \ldots, \dot{q}_m$ to new variables $s_{l+1}, s_{l+2}, \ldots, s_n$ and $p_{k+1}, p_{k+2}, \ldots, p_m$ by using the following formulas

$$s_i = \frac{\partial L}{\partial q_i} \quad (i = l+1, \ldots, n), \quad p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (i = k+1, \ldots, m), \tag{2.1}$$

assuming that $0 \leq k \leq l \leq m \leq n$. If the Jacobian of the transformation is not equal to zero, /18

$$\frac{D\left(\frac{\partial L}{\partial q_{l+1}}, \ldots, \frac{\partial L}{\partial q_n}, \frac{\partial L}{\partial \dot{q}_{k+1}}, \ldots, \frac{\partial L}{\partial \dot{q}_m}\right)}{D(q_{l+1}, \ldots q_n, \dot{q}_{k+1}, \ldots, \dot{q}_m)} \neq 0, \tag{2.2}$$

then equations (2.1) can be solved for the transforms of Lagrangian variables, The quantities p_i are referred to as "generalized moments"; the quantities s_i, which, by virtue of equation (1.18), coincide with the corresponding values of p_i, we shall refer to as "moments of the second order".

From equation (2.1) it is evident that for this particular transformation it is characteristic that the state of motion of this system, in the case of one group of degrees of freedom, is determined by generalized coordinates and velocities, in the case of the second group by generalized coordinates and moments, and, in the case of the last group by generalized moments and their derivatives.

Let us consider the function M*, which is defined by the following relationship:

$$M^* = -L + \sum_{i=k+1}^{m} p_i \dot{q}_i + \sum_{i=l+1}^{n} s_i q_i. \tag{2.3}$$

Solving equations (2.1) for the old variables, which is possible on the basis of the condition of (2.2), and substituting in (2.3) the appropriate values of q_i and $\dot{q}_i$ with p_i and s_i, we arrive at the following:

$$M(q_1, \ldots, q_l, \dot{q}_1, \ldots, \dot{q}_k, p_{k+1}, \ldots, p_m, s_{l+1}, \ldots, s_n, \dot{q}_{m+1}, \ldots, \dot{q}_n) = M^*_{\substack{\dot{q}_{k+1}, \ldots, \dot{q}_m \to p_{k+1}, \ldots, p_m \\ q_{l+1}, \ldots, q_n \to s_{l+1}, \ldots, s_n}} \tag{2.4}$$

The total variation of this function will be as follows:

$$\delta M = \sum_{i=1}^{l} \frac{\partial M}{\partial q_i} \delta q_i + \sum_{i=m+1}^{n} \frac{\partial M}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=1}^{k} \frac{\partial M}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=k+1}^{m} \frac{\partial M}{\partial p_i} \delta p_i + \sum_{i=l+1}^{n} \frac{\partial M}{\partial s_i} \delta s_i. \tag{2.5}$$

On the other hand, varying the explicit expression for M, obtained from equations (2.3) and (2.1), we arrive at the following: /19

$$\delta M = -\sum_{i=1} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) + \sum_{i=k+1}^{m} (p_i \delta \dot{q}_i + \dot{q}_i \delta p_i) + \sum_{i=l+1}^{n} (s_i \delta q_i + q_i \delta s_i). \tag{2.6}$$

Comparing equations (2.5) and (2.6) and making allowance for the transformation formula (2.1), in view of the arbitrariness of the variations which enter into equations (2.5) and (2.6), we arrive at:

$$\left.\begin{aligned} &\frac{\partial M}{\partial q_i} = -\frac{\partial L}{\partial q_i} (i = 1, 2, \ldots, l), \frac{\partial M}{\partial p_i} = \dot{q}_i (i = k+1, \ldots, m), \\ &\frac{\partial M}{\partial \dot{q}_i} = -\frac{\partial L}{\partial \dot{q}_i} (i = 1, 2, \ldots, k, m+1, \ldots, n), \frac{\partial M}{\partial s_i} = q_i \\ &\qquad\qquad (i = l+1, \ldots, n). \end{aligned}\right\} \tag{2.7}$$

In this case the differential equations of motion will have the following form:

$$\frac{d}{dt}\left(\frac{\partial M}{\partial \dot{q}_i}\right) - \frac{\partial M}{\partial q_i} = 0 \quad (i = 1, 2, \ldots, k); \tag{2.8}$$

$$\frac{d}{dt}\left(\frac{\partial M}{\partial s_i}\right) - \frac{\partial M}{\partial p_i} = 0 \quad (i = l+1, \ldots, m); \tag{2.9}$$

$$\frac{dq_i}{dt} = \frac{\partial M}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial M}{\partial q_i} \quad (i = k+1, \ldots, l), \tag{2.10}$$

$$p_i = \frac{\partial M}{\partial \dot{q}_i}, \quad q_i = \frac{\partial M}{\partial s_i} \quad (i = m+1, \ldots, n). \tag{2.11}$$

But since $s_i = p_i$ then the subsystem (2.9) can be written as follows:

$$\frac{d}{dt}\left(\frac{\partial M}{\partial \dot{p}_i}\right) - \frac{\partial M}{\partial p_i} = 0 \quad (i = l+1, \ldots, m), \tag{2.12}$$

while the subsystem (2.11) will have the following form:

$$\dot{p}_i = \frac{d}{dt}\left(\frac{\partial M}{\partial \dot{q}_i}\right), \quad \dot{q}_i = -\frac{d}{dt}\left(\frac{\partial M}{\partial \dot{p}_i}\right) \quad (i = m+1,\ldots,n). \tag{2.13}$$

Thus, the motion of the mechanical system will be described by the system of equations (2.8), (2.10), (2.12) and (2.13). /20

If, within transformation (2.1), k = 0 and ℓ = n, then the equations of motion will assume a Hamiltonian form (2.10), and function M in this case will be a characteristic Hamiltonian function, which has traditionally been designated by the symbol H:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2, \ldots, n), \tag{2.14}$$

the Hamiltonian H being defined by this formula:

$$H = -L + \sum_{i=1}^{n} p_i \dot{q}_i. \tag{2.15}$$

If ℓ = m = n in the transformation (2.1), then the equations of motion will consist only of the subsystems (2.8) and (2.10), and, consequently, they will be of mixed Lagrangian-Hamiltonian form:

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\partial M}{\partial \dot{q}_i}\right) - \frac{\partial M}{\partial q_i} = 0 \quad (i = 1, 2, \ldots, k), \\ &\frac{dq_i}{dt} = \frac{\partial M}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial M}{\partial q_i} \quad (i = k+1, \ldots, n). \end{aligned}\right\} \tag{2.16}$$

The equations of motion (2.16) we shall refer to as the Routh equations. Similarly, the characteristic function of equations (2.16) we shall refer to as Routhian, which, according to (2.3), will have the following form:

$$M = -L + \sum_{i=k+1}^{n\ell} p_i \dot{q}_i, \tag{2.17}$$

where the transformed variables, with the help of the transformation formulas, must be expressed in terms of new variables.

The Routh equations are found to be convenient in the study of the rectilinear-rotational motion of artificial heavenly bodies. In this case, in lieu of the canonical variables q_i, p_i, which enter into the subsystem of Hamiltonian equations, it is expedient to adopt the osculating Keplerian elements of the orbit of the center of masses of the spaceship, which vary slowly with the respect to time. Similarly, in place of the generalized coordinates for the subsystem of Lagrangian equations, it is expedient to

adopt Eulerian angles.

If $l = 0$, $m = n$, then the equations of motion will have a Lagrangian form; however the function will be expressed in terms of first-order and second-order moments. This characteristic function was pointed out earlier by V.M. Tatevskiy (8, 9]. The case $m = 0$ was also pointed out by V.M. Tatevskiy [9] and subsequently studied in detail by K. M. Raitzin [10, 11]. New forms of the equations of dynamics can be obtained for still other limitations of k, l, m and n.

Here let us note still another partial case, which was obtained for $k = l \leq m = n$. The equations of motion are then written in the following form:

$$\frac{d}{dt}\left(\frac{\partial M_1}{\partial \dot{p}_{1i}}\right) - \frac{\partial M_1}{\partial p_{1i}} = 0 \quad (i = 1, 2, \ldots, n), \tag{2.18}$$

where $M_1 = M$, $p_{1i} = p_i$.

We shall further assume that

$$p_{2i} = \frac{\partial M}{\partial \dot{p}_{1i}}, \quad \dot{p}_{2i} = \frac{\partial M}{\partial p_{1i}} \quad (i = 1, 2, \ldots, n), \tag{2.19}$$

assuming that the Jacobian of the transformation (2.19) is not equal to zero. The quantities p_{2i} are referred to as "moments of the third order". Then assuming that:

$$M_2^* = M_1 - \sum_{i=1}^{n}\left(\frac{\partial M_1}{\partial \dot{p}_{1i}}\dot{p}_{1i} + \frac{\partial M_1}{\partial p_{1i}}p_{1i}\right), \tag{2.20}$$

we arrive at the equations of motion in the following form, following certain simple transformations:

$$\frac{d}{dt}\left(\frac{\partial M_2}{\partial \dot{p}_{2i}}\right) - \frac{\partial M_2}{\partial p_{2i}} = 0 \qquad (i = 1, 2, \ldots, n), \tag{2.21}$$

where M_2 denotes the result of substitution of p_{2i} and $\dot{p}_{2i}$ from formula (2.19), in place of $\dot{p}_{1i}$ and p_{1i}, in equation (2.20).

Transformations such as that appearing in (2.19) can be repeated a number of times, provided each time that the corresponding Jacobians are not equal to zero. As a result we obtain a Lagrangian system of equations:

$$\frac{d}{dt}\left(\frac{\partial M_s}{\partial \dot{p}_{si}}\right) - \frac{\partial M_s}{\partial p_{si}} = 0 \qquad (i = 1, 2, \ldots, n), \tag{2.22}$$

in which by the symbol p_{si} we denote the moments of the s + 1-st order. These /22

moments and their derivatives $\dot{p}_{si}$ are defined with the help of the following formulas:

$$p_{si}=\frac{\partial M_{s-1}}{\partial \dot{p}_{s-1,\,i}},\quad \dot{p}_{si}=\frac{\partial M_{s-1}}{\partial p_{s-1,\,i}}. \tag{2.23}$$

For the new Lagrangian we obtain:

$$M_s=\left\{M_{s-1}-\sum_{i=1}^{n}\left(\dot{p}_{si}\dot{p}_{s-1,\,i}+p_{si}p_{s-1,\,i}\right)\right\}_{p_{s-1,\,i},\ \dot{p}_{s-1,\,i}\to p_{si},\ \dot{p}_{si}}. \tag{2.24}$$

The invariance of the Lagrangian form of equations, in the case of transition to higher order moments, for the problem of small oscillations, provided that instead of generalized coordinates we make use of normal coordinates, for in this case the higher order moments are proportional to the derivatives (with respect to time) of the generalized coordinates of the same order. In this respect, the Lagrangian form of equations in space moments ıof higher orders is a consequence of the variational nature of the laws of ımechanics.

To the equation (2.22) corresponds the principle of least action in the form:

$$\delta\int_{t_0}^{t}M_s\left(p_{si},\ \dot{p}_{si}\right)dt=0. \tag{2.25}$$

§3. Differential Equations of Motion of Material Point in Curvilinear Coordinates

We consider here the problem of the movement of a free material point within a potential field of forces, and obtain the equations of motion in the form which is most useful in celestial ballistics, on the basis of orthogonal curvilinear coordinates.

Let us assume that the rectangular Cartesian coordinates of points x, y, z are expressed in terms of new mutually independent variables q_1, q_2, q_3 with the help of single-valued relationships:

$$\left.\begin{aligned}x&=x\left(q_1,q_2,q_3\right),\\ y&=y\left(q_1,q_2,q_3\right),\\ z&=z\left(q_1,q_2,q_3\right).\end{aligned}\right\} \tag{3.1}$$

Let us assume that the reverse transformation: /23

$$\left.\begin{aligned}q_1&=q_1\left(x,y,z\right),\\ q_2&=q_2\left(x,y,z\right),\\ q_3&=q_3\left(x,y,z\right)\end{aligned}\right\} \tag{3.2}$$

is also single-valued.

Such variables as q_1, q_2, q_3 form a system of curvilinear coordinates. The particular forms of curvilinear coordinate systems are the cylindrical, the spherical, the ellipsoidal, and the paraboloidal coordinates.

Having made some choice of a curvilinear coordinate system and considered one of the equations of (3.2) $q_i = q_i(x, y, z)$, we find that a definite surface which is called a coordinate surface, corresponds to the equation $q_i = \text{const}$ in space.

Each point in space is defined by the intersection of three coordinate surfaces:

$$q_i = c_i \quad (i = 1, 2, 3). \tag{3.3}$$

The intersection of two coordinate surfaces determines a coordinate line. Along a coordinate line any two curvilinear coordinates maintain constant values.

Through every point in space there pass three coordinate lines (Figure 3): line q_1, on which the coordinates q_2 and q_3 are constant, line q_2, along which the coordinates q_1 and q_3 are constant, and, finally, line q_3, along which the coordinates q_1 and q_2 remain constant.

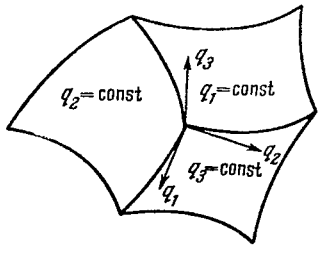

Figure 3.

The tangents to coordinate lines may be referred to as "axes of curvilinear coordinates". A system of curvilinear coordinates will be orthogonal provided that at any point in space the coordinate axes are mutually perpendicular. It is not difficult to establish that the condition of orthogonality of a system of curvilinear coordinates can be described in the following form: /24

$$\left.\begin{aligned}
\frac{\partial x}{\partial q_1}\frac{\partial x}{\partial q_2}+\frac{\partial y}{\partial q_1}\frac{\partial y}{\partial q_2}+\frac{\partial z}{\partial q_1}\frac{\partial z}{\partial q_2}=0,\\
\frac{\partial x}{\partial q_2}\frac{\partial x}{\partial q_3}+\frac{\partial y}{\partial q_2}\frac{\partial y}{\partial q_3}+\frac{\partial z}{\partial q_2}\frac{\partial z}{\partial q_3}=0,\\
\frac{\partial x}{\partial q_3}\frac{\partial x}{\partial q_1}+\frac{\partial y}{\partial q_3}\frac{\partial y}{\partial q_1}+\frac{\partial z}{\partial q_3}\frac{\partial z}{\partial q_1}=0.
\end{aligned}\right\} \tag{3.4}$$

In compiling the Lagrangian equations it is first of all necessary to express the kinetic energy of a material point in terms of generalized velocities. For this purpose it is expedient to calculate the square of the linear element ds. With the help of equation (3.1) we obtain the following:

$$ds^2 = H_1^2\,dq_1^2 + H_2^2 dq_2^2 + H_3^2\,dq_3^2, \tag{3.5}$$

where H_1, H_2 and H_3 are the Lamé coefficients, defined by the following formulas:

$$H_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2} \quad (i = 1, 2, 3). \tag{3.6}$$

With the help of equation (3.5) we come to the following expression for the square of the velocity of the point:

$$v^2 = H_1^2\dot{q}_1^2 + H_2^2\dot{q}_2^2 + H_3^2\dot{q}_3^2. \tag{3.6'}$$

Since in the case of the great majority of problems encountered in celestial ballistics the force function is proportional to the mass of the moving point, this mass does not appear in the Lagrangian equations of motion. Therefore, by U and L we shall understand, respectively, the force function and the Lagrangian, as reduced to the mass of the moving point.

Keeping in mind equation (3.6), we arrive at the following expression for the Lagrangian:

$$L = \frac{1}{2}(H_1^2\dot{q}_1^2 + H_2^2\dot{q}_2^2 + H_3^2\dot{q}_3^2) + U(q_1, q_2, q_3, t). \tag{3.7}$$

Cylindrical Coordinates. The rectangular coordinates are expressed in terms of cylindrical coordinates by means of the following formulas:

$$\left.\begin{aligned} x &= \rho\cos\lambda, \\ y &= \rho\sin\lambda, \\ z &= z, \end{aligned}\right\} \tag{3.8}$$

while for the reverse transformation we have /25

$$\left.\begin{aligned} \rho &= \sqrt{x^2 + y^2}\,, \\ \lambda &= \arctan\frac{y}{x}\,, \\ z &= z. \end{aligned}\right\} \tag{3.9}$$

For cylindrical coordinates (Figure 4) the coordinate surfaces consist of a cylinder passing through the point in space under consideration (the axis of this cylinder coincides with the z-axis), the half-plane λ = const passing through the given point M and through the z axis, and, finally the plane z = = const, which is parallel to the basic coordinate plane. The coordinate

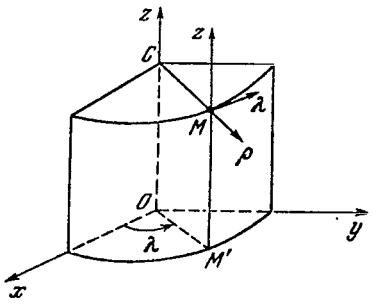

Figure 4

lines consist of the straight line CM, along which λ and z are constant; the arc of the circle BM, on which ρ and z are constant; and the element of the cylinder MM', on which ρ and λ are constant. It is not difficult to verify, in this case, that the conditions of (3.4) are met, and that, consequently, the system of coordinates is orthogonal.

With the help of formulas (3.6) and (3.8) we determine the Lamé coefficients:

$$\left.\begin{aligned} H_\rho &= 1, \\ H_\lambda &= \rho, \\ H_z &= 1. \end{aligned}\right\} \tag{3.10}$$

Then, on the basis of equation (3.5) for a linear element we obtain the following expression:

$$ds^2 = d\rho^2 + \rho^2\, d\lambda^2 + dz^2, \tag{3.11}$$

while the Lagrangian (3.7) assumes the following form:

$$L = \frac{1}{2}(\dot\rho^2 + \rho^2\dot\lambda^2 + \dot z^2) + U(\rho, \lambda, z, t). \tag{3.12}$$

The equations of motion (1.18) in cylindrical coordinates are written in the following form: /26

$$\left.\begin{aligned} \ddot\rho - \rho\dot\lambda^2 &= U'_\rho, \\ \frac{d}{dt}(\rho^2\dot\lambda) &= U'_\lambda, \\ \ddot z &= U'_z. \end{aligned}\right\} \tag{3.13}$$

Spherical Coordinates. The rectangular coordinates are expressed in terms of spherical coordinates in the following manner:

$$\left.\begin{aligned} x &= r\cos\varphi\cos\lambda, \\ y &= r\cos\varphi\sin\lambda, \\ z &= r\sin\varphi. \end{aligned}\right\} \tag{3.14}$$

The geometrical meaning of the spherical coordinates in this case is evident (Figure 5). Here the angle ϕ is counted within the limits of $-\pi/2$ and

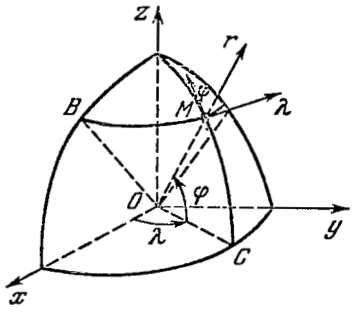

Figure 5

$\pi/2$; the angle λ is counted within the limits of 0 and 2π; and r is considered to be invariable from 0 to $+\infty$.

A geocentric system of coordinates is usually employed in dealing with the motion of artificial earth satellites. In such a system the origin is at the center of inertia of the earth, and the basic plane consists of the earth's equatorial plane. In this case the coordinate ϕ becomes the geographical latitude of the satellite. In addition, if the system of coordinates is rigidly bound with the earth, and if the x axis lies in the plane of the Greenwich meridian, then the coordinate λ becomes the geographical longitude of the satellite. In an inertial coordinate system whose x axis is directed toward the vernal equinox, λ will be the right ascension.

The formulas used for transforming rectangular coordinates into spherical coordinates are as follows: /27

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2+z^2}, \\ \lambda &= \arctan\frac{y}{x}, \\ \varphi &= \arctan\frac{z}{\sqrt{x^2+y^2}}. \end{aligned}\right\} \tag{3.15}$$

In a spherical coordinate system the coordinate surfaces consists of the following: the sphere r = const, the circular cone ϕ = const, and the half-plane λ = const. The coordinate lines consist of the straight line OM, along which the coordinates ϕ and λ are constant, the arc of meridian CM, on which r and λ are constant, and the arc of parallel BM, on which the coordinates r and ϕ are constant.

From formulas (3.4) and (3.14) it follows that a system of spherical coordinates is orthogonal.

Calculating the Lamé coefficients with the help of formulas (3.6) and (3.14), we find that

$$H_r = 1,\ H_\varphi = r,\ H_\lambda = r\cos\varphi, \tag{3.16}$$

for the square of the linear element we obtain the following expression:

$$ds^2 = dr^2 + r^2 d\varphi^2 + r^2\cos^2\varphi\ d\lambda^2. \tag{3.17}$$

The Lagrangian function will assume the following form:

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 + r^2\cos^2\varphi\cdot\dot{\lambda}^2) + U(r, \varphi, \lambda), \tag{3.18}$$

from which, according to (1.18) we obtain the Lagrangian equations of motion:

$$\left.\begin{aligned} \ddot{r} - r\dot{\varphi}^2 - r\cos^2\varphi\cdot\dot{\lambda}^2 &= U'_r, \\ \frac{d}{dt}(r^2\cos^2\varphi\cdot\dot{\lambda}) &= U'_\lambda, \\ \frac{d}{dt}(r^2\dot{\varphi}) + r^2\dot{\lambda}^2\sin\varphi\cos\varphi &= U'_\varphi. \end{aligned}\right\} \tag{3.19}$$

Ellipsoidal coordinates have been widely used in the problem of the satellites of a spheroidal planet. Below we consider two types of degenerate ellipsoidal coordinates.

Oblate Spheroidal Coordinates. These coordinates are associated with /28
rectangular coordinates by means of the following formulas [1]:

$$\left.\begin{aligned} x &= c\,\mathrm{ch}\,\psi\cos\vartheta\cos\lambda, \\ y &= c\,\mathrm{ch}\,\psi\cos\vartheta\sin\lambda, \\ z &= c\,\mathrm{sh}\,\psi\sin\vartheta, \end{aligned}\right\} \tag{3.20}$$

where c is a constant multiplier having the dimensionality of length. With respect to ψ, ϑ, λ we shall assume that they satisfy the following conditions:

$$0 \leqslant \psi < +\infty, \quad -\frac{\pi}{2} \leqslant \vartheta \leqslant \frac{\pi}{2}, \quad 0 \leqslant \lambda \leqslant 2\pi. \tag{3.21}$$

Under these conditions, to every point in space there corresponds a unique combination of values of ψ, λ, ϑ.

Combining equations (3.20), with ψ = const, we discover that the point must be located upon an ellipsoid of rotation:

$$\frac{x^2 + y^2}{c^2\,\mathrm{ch}^2\psi} + \frac{z^2}{c^2\,\mathrm{sh}^2\psi} = 1. \tag{3.22}$$

with major semiaxis c ch ψ and minor semiaxis c sh ψ. The ellipsoid of (3.22) is an oblate ellipsoid of rotation whose axis of symmetry coincides with the z axis.

For ϑ = const, it follows from (3.20) that the point is found upon a single-sheet hyperboloid of rotation:

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[1] Oblate spheroidal coordinates are associated with cylindrical coordinates by means of the following relationship: $z + i\rho = ic\,\mathrm{ch}\,(\lambda - i\phi)$.

$$\frac{x^2+y^2}{c^2\cos^2\vartheta}-\frac{z^2}{c^2\sin^2\vartheta}=1. \tag{3.23}$$

Finally, for λ = const, the point lies on the plane

$$y - x \tan \lambda = 0. \tag{3.24}$$

The surfaces of (3.22) - (3.24) are coordinate surfaces (Figure 6).

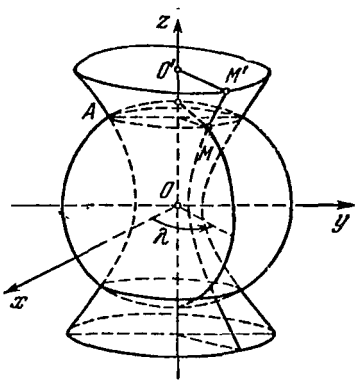

Figure 6

The coordinate lines in the spheroidal system are as follows: the hyperbola MM', on which λ and θ are constant; the parallel of the spheroid AM, for which the corresponding values of ψ and θ are constant; and the elliptical arc of meridian of the spheroid which passes through the point M. From equations (3.4) and (3.20) it follows that oblate spheroidal coordinates are orthogonal: that is, the coordinate surfaces are an hyperboloid, and ellipsoid, and a plane, which intersect one another at right angles. /29

Such a coordinate system has been used in the new theory of the motion of artificial earth satellites [14].

The differentiating the transformation formula (3.20) with respect to the spheroidal coordinates ψ, ϑ and λ, and substituting the values found in formulas (3.6), we arrive at the Lamé coefficients:

$$\begin{aligned} H_\psi = H_\vartheta = c\sqrt{\operatorname{ch}^2\psi-\cos^2\vartheta}, \\ H_\lambda = c\operatorname{ch}\psi\cos\vartheta. \end{aligned} \tag{3.25}$$

Substituting the values found for the Lamé coefficients in formula (3.5), we then arrive at the following value for the linear element ds:

$$ds^2 = c^2(\operatorname{ch}^2\psi-\cos^2\vartheta)(d\psi^2+d\vartheta^2)+c^2\operatorname{ch}^2\psi\cos^2\vartheta\cdot d\lambda^2. \tag{3.26}$$

Making allowance for equations (3.7) and (3.26) we then have the following value for the Lagrangian function:

$$L=\frac{c^2}{2}[J(\dot\psi^2+\dot\vartheta^2)+\operatorname{ch}^2\psi\cos^2\vartheta\cdot\dot\lambda^2]+U(\psi,\vartheta,\lambda), \tag{3.27}$$

where $J = \operatorname{ch}^2\psi - \cos^2\vartheta$, which leads to the following system of differential equations of motion:

$$\left.\begin{aligned}
&\frac{d}{dt}(J\dot{\psi})+\frac{1}{2}\operatorname{sh}2\psi\cdot(\dot{\psi}^2+\dot{\vartheta}^2)-\frac{1}{2}\operatorname{sh}2\psi\cos^2\vartheta\cdot\dot{\lambda}^2=\frac{1}{c^2}U'_{\psi},\\
&\frac{d}{dt}(J\dot{\vartheta})+\frac{1}{2}\sin 2\vartheta\cdot(\dot{\psi}^2+\dot{\vartheta}^2)+\frac{1}{2}\operatorname{ch}^2\psi\sin 2\vartheta\cdot\dot{\lambda}^2=\frac{1}{c^2}U'_{\vartheta},\\
&\frac{d}{dt}(\operatorname{ch}^2\psi\cos^2\vartheta\cdot\dot{\lambda}^2)=\frac{1}{c^2}U'_{\lambda}.
\end{aligned}\right\} \qquad (3.28)$$

Prolate Spheroidal Coordinates. We now take up the second system of degenerate ellipsoidal coordinates which is frequently used in the classical problem of celestial mechanics of two immobile centers [15]. /30

Prolate spheroidal coordinates u, v, w are associated with rectangular coordinates by the following relationships:

$$\left.\begin{aligned}
x&=c\operatorname{ch}v\cos u,\\
y&=c\operatorname{sh}v\sin u\sin w,\\
z&=c\operatorname{sh}v\sin u\cos w,
\end{aligned}\right\} \qquad (3.29)$$

where c is a certain constant quantity having the dimensionality of length. We shall assume that the spheroidal coordinates satisfy the following conditions:

$$0\leqslant v<+\infty,\quad -\frac{\pi}{2}\leqslant u\leqslant\frac{\pi}{2},\quad 0\leqslant w\leqslant 2\pi. \qquad (3.30)$$

If the conditions of (3.30) are met, then for every point in space there exists a unique combination of values of u, v, w.

To the system of prolate spheroidal coordinates correspond the following coordinate surfaces: for v = const we have a prolate ellipsoid of rotation,

$$\frac{x^2}{c^2\operatorname{ch}^2v}+\frac{y^2+z^2}{c^2\operatorname{sh}^2v}=1, \qquad (3.31)$$

whose axis of symmetry coincides with the x axis, while the foci are located at the points (±c, 0, 0); for u = const which is cofocal with the two-cavity hyperboloid of rotation

$$\frac{x^2}{c^2\cos^2u}-\frac{y^2+z^2}{c^2\sin^2u}=1 \qquad (3.32)$$

and for w = const, we have the plane

$$y - z \tan w = 0. \qquad (3.33)$$

The coordinate surfaces and lines are illustrated in Figure 7. The prolate spheroidal coordinates are also orthogonal. /31

From simple calculations we obtain the following values of the Lamé coefficients:

$$\left.\begin{aligned}
H_u&=H_v=c\sqrt{\operatorname{ch}^2v-\cos^2u},\\
H_w&=c\operatorname{sh}v\sin u.
\end{aligned}\right\} \qquad (3.34)$$

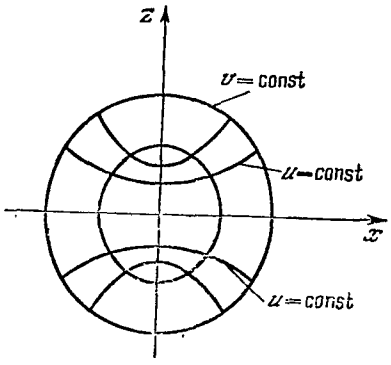

Figure 7

Then we obtain the following expression for the linear element:

$$ds^2 = c^2[J\,(du^2 + dv^2) + \mathrm{sh}^2 v \sin^2 u \cdot dw^2], \qquad (3.35)$$

while the Lagrangian function assumes the following form:

$$L = \frac{c^2}{2}[J\,(\dot{u}^2 + \dot{v}^2) + \mathrm{sh}^2 v \sin^2 u \cdot \dot{w}^2] + U(u, v, w, t), \qquad (3.36)$$

where the following designation is used

$$J = \mathrm{ch}^2 v - \cos^2 u. \qquad (3.37)$$

The equations of motion of a point in prolate spheroidal coordinates are described as follows:

$$\left.\begin{aligned} &\frac{d}{dt}(J\dot{u}) - \sin u \cdot \cos u\,(\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \mathrm{sh}^2 v) = \frac{1}{c^2} U'_u, \\ &\frac{d}{dt}(J\dot{v}) - \mathrm{sh}\, v \cdot \mathrm{ch}\, v\,(\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \sin^2 u) = \frac{1}{c^2} U'_v, \\ &\frac{d}{dt}(\mathrm{sh}^2 v \sin^2 u \cdot \dot{w}) = \frac{1}{c^2} U'_w. \end{aligned}\right\} \qquad (3.38)$$

Paraboloidal Coordinates. In dealing with the problem of the motion of a spaceship within a central Newtonian field of force, and with constant vector of reaction acceleration [16] and [17, 18], and also in studying the problem of the motion of artificial earth satellites with allowance for light pressure, the scientists make use of paraboloidal coordinates. Such coordinates are associated with rectangular coordinates by the following relationships:

$$\left.\begin{aligned} x &= \frac{1}{2}(\xi^2 - \eta^2), \\ y &= \xi\eta\cos\varphi, \\ z &= \xi\eta\sin\varphi. \end{aligned}\right\} \qquad (3.39)$$

From the last two equations (3.39) we find $y^2 + z^2 = \xi^2\eta^2$. Combining this with x^2, we have /32

$$x^2 + y^2 + z^2 = \frac{1}{4}(\xi^2 + \eta^2)^2,$$

from which ξ = const we obtain the following equation of the coordinate surface:

$$y^2 + z^2 = \xi^4 - 2x\xi^2. \tag{3.40}$$

Equation (3.40) defines the paraboloid of rotation with focus at the origin of coordinates and with axis coinciding with the negative portion of the x axis.

In analogous fashion we find the second coordinate surface:

$$y^2 + z^2 = \eta^4 + 2x\eta^2, \tag{3.41}$$

which also is a paraboloid with focus at the origin of coordinates and with axis lying along the x axis.

The third coordinate surface is the plane which passes through the x axis:

$$z - y \tan \phi = 0. \tag{3.42}$$

The coordinate lines are as follows:

line ξ is the parabola

$$y^2 = \eta^4 \cos^2 \varphi + 2\eta^2 x \cos^2 \varphi,$$
$$z = y \tan \varphi,$$

line η is the parabola

$$y^2 = \xi^4 \cos^2 \varphi - 2\eta^2 x \cos^2 \varphi,$$
$$z = y \tan \rho,$$

line ϕ is the circle

$$y^2 + z^2 = \xi^2\eta^2, \quad x = \frac{1}{2}(\xi^2 - \eta^2).$$

The relative positions of the coordinate planes and lines are shown in Figure 8. The paraboloidal coordinates are orthogonal, as is evident from equations (3.4) and (3.39).

Formulas (3.6) and (3.39) lead to the following expressions for the Lamé coefficients: /33

$$H_\xi = H_\eta = \sqrt{\xi^2 + \eta^2}, \qquad H_\varphi = \xi\eta. \tag{3.43}$$

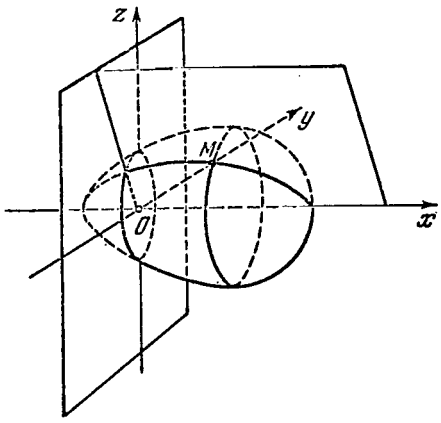

Figure 8

Then the square of the linear element is determined from the formula:

$$ds^2 = (\xi^2 + \eta^2)(d\xi^2 + d\eta^2) + \xi\eta\, d\varphi^2, \qquad (3.44)$$

and the Lagrangian will be:

$$L = \frac{1}{2}[(\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2) + \xi\eta\dot{\varphi}^2] + U(\xi, \eta, \varphi, t). \qquad (3.45)$$

The Lagrangian equation of motion in paraboloidal coordinates will be written as follows:

$$\left.\begin{aligned} &\frac{d}{dt}[(\xi^2 + \eta^2)\dot{\xi}] - [\xi(\dot{\xi}^2 + \dot{\eta}^2) + \eta\dot{\varphi}^2] = U'_\xi, \\ &\frac{d}{dt}[(\xi^2 + \eta^2)\dot{\eta}] - [\eta(\dot{\xi}^2 + \dot{\eta}^2) + \xi\dot{\varphi}^2] = U'_\eta, \\ &\frac{d}{dt}(\xi\eta\dot{\varphi}) = U'_\varphi. \end{aligned}\right\} \qquad (3.46)$$

NOTE: If both foci are removed to infinity, then the ellipsoidal coordinates degenerate into rectangular coordinates; if only one focus is removed to infinity the ellipsoidal coordinates become paraboloidal; and, finally, when the two foci coincide, the ellipsoidal coordinates degenerate into spherical coordinates.

§ 4. Canonical Transformations. Jacobi's Theorem.

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According to the results of § 2 the motion of a dynamic system can be formulated in a system of equations similar to the Routh equations:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad (i = 1, 2, \ldots, k \leqslant n), \qquad (4.1)$$

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{q}_j}\right) - \frac{\partial H}{\partial q_j} = 0 \quad (j = k+1, \ldots, n); \qquad (4.2)$$

in this system of equations the characteristic function H, which is defined by formula (2.17), will in the general sense, be independent of the quantities $t, q_1, \ldots, q_n, p_1, \ldots, p_k, q_{k+1}, \ldots, \dot{q}_n$. The variables p_i and q_i of the subsystem of equations (4.1) we shall refer to as "conjugate canonical variables" (q_i is the generalized coordinate, p_i is the generalized moment).

Let us examine those transformations of the canonical variables q_i and p_i which do not disturb the Routhian form of equations (4.1) and (4.2). These transformations we shall refer to as "canonical".

Tranformation of the old variables q_i and p_i to the new variables ξ_i and η_i we shall define with the function

$$V(t, q_1, \ldots, q_k, \xi_1, \ldots, \xi_k),$$

this will be referred to as the "generating function". The transformation formulas will have the form

$$cp_i = \frac{\partial V}{\partial q_i}, \qquad \eta_i = -\frac{\partial V}{\partial \xi_i} \qquad (i = 1, 2, \ldots, k), \tag{4.3}$$

where c is a constant (the valency of the transformation).

From equations (4.1) and (4.3) it follows that:

$$\frac{\partial H}{\partial \eta_i} = \sum_{j=1}^{k} \left(\dot{q}_j \frac{\partial p_j}{\partial \eta_i} - \dot{p}_j \frac{\partial q_j}{\partial \eta_i} \right),$$

$$\frac{\partial p_j}{\partial \eta_i} = \frac{1}{c} \sum_{s=1}^{k} \frac{\partial^2 V}{\partial q_j \partial q_s} \frac{\partial q_s}{\partial \eta_i}$$

and /35

$$c\dot{p}_s = \frac{\partial^2 V}{\partial q_s \partial t} + \sum_{j=1}^{k} \left(\frac{\partial^2 V}{\partial q_s \partial q_j} \dot{q}_j + \frac{\partial^2 V}{\partial q_s \partial \xi_j} \dot{\xi}_j \right),$$

therefore,

$$\frac{\partial H}{\partial \eta_i} = -\frac{1}{c} \sum_{s=1}^{k} \frac{\partial q_s}{\partial \eta_i} \left[\frac{\partial q_s}{\partial \eta_j} + \sum_{j=1}^{k} \frac{\partial^2 V}{\partial q_s \partial \xi_j} \dot{\xi}_j \right].$$

Taking into consideration the fact that

$$\sum_{s=1}^{k} \frac{\partial^2 V}{\partial q_i \partial \xi_j} \frac{\partial q_s}{\partial \eta_i} = \begin{cases} 0 \text{ if } i \neq j, \\ -1 \text{ if } i = j, \end{cases}$$

we find that $\frac{\partial H}{\partial \eta_i} = -\frac{1}{c} \frac{\partial}{\partial \eta_i} \left(\frac{\partial V}{\partial t} \right) + \frac{1}{c} \dot{\xi}_i$, or $\dot{\xi}_i = \frac{\partial}{\partial \eta_i} \left(cH + \frac{\partial V}{\partial t} \right)$. It can also be

be shown that $\dot{\eta}_i = -\frac{\partial}{\partial \xi_i} \left(cH + \frac{\partial V}{\partial t} \right)$.

Let us assume that

$$K = cH + \frac{\partial V}{\partial t}. \tag{4.4}$$

Since

Since

$$\frac{\partial K}{\partial \dot{q}_i} = c\frac{\partial H}{\partial \dot{q}_i}, \qquad \frac{\partial K}{\partial q_i} = c\frac{\partial H}{\partial q_i},$$

then, instead of the system (4.1) - (4.2), we finally have the following:

$$\dot{\xi}_i = \frac{\partial K}{\partial \eta_i}, \quad \dot{\eta}_i = -\frac{\partial K}{\partial \xi_i} \qquad (i = 1, 2, \ldots, k), \tag{4.5}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_j}\right) - \frac{\partial K}{\partial q_j} = 0 \qquad (j = k+1, \ldots, n). \tag{4.6}$$

The transformation (4.3) is possible only when the Jacobian

$$\det\left(\frac{\partial^2 V}{\partial q_i \partial \xi_j}\right)$$

is not equal to zero.

We are now able to formulate a theorem.

Theorem. If the canonical variables q_i and p_i are transformed into new variables ξ_i and η_i, with the aid of the generating function V (t, q_1, q_2,..., q_k, ξ_1, ξ_2,..., ξ_k), then the differeintial equations for the new variables will be of Routhian form, and the new characteristic Routh function will resemble (4.4). /36

From this theorem we can derive a theorem for the transformation of Hamiltonian systems for the case k = n.

Using similar arguments, we can show the correctness of the following theorem.

Theorem. Let the variables q_i and p_i be transformed into new variables with the help of the formulas:

$$\left.\begin{aligned} &\frac{\partial V}{\partial q_i} = cp_i \ (i = 1, 2, \ldots, l), \quad \frac{\partial V}{\partial p_j} = -cq_j \ (j = l+1, \ldots, k), \\ &\frac{\partial V}{\partial \xi_r} = -\eta_r \ (r = 1, 2, \ldots, m), \quad \frac{\partial V}{\partial \eta_s} = \xi_s \ (s = m+1, \ldots, k), \end{aligned}\right\} \tag{4.7}$$

in which the generating function V assumes the form:

$$V(t, q_1, q_2, \ldots, q_l, p_{l+1}, \ldots, p_k, \xi_1, \ldots, \xi_m, \eta_{m+1}, \ldots, \eta_k);$$

then the Routhian equations of motion (4.1) - (4.2) preserve their form

$$\left.\begin{aligned} &\frac{d\xi_i}{dt} = \frac{\partial K}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{\partial K}{\partial \xi_i} \quad (i = 1, 2, \ldots, k), \\ &\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_j}\right) - \frac{\partial K}{\partial q_j} = 0 \quad (j = k+1, \ldots, n), \end{aligned}\right\} \tag{4.8}$$

while the new characteristic Routh function K is defined by

$$K = cH + \frac{\partial V}{\partial t}. \tag{4.9}$$

From these two systems we can deduce the following corollaries relating to the pure Hamiltonian systems of (2.14):

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2, \ldots, n). \tag{4.10}$$

Transformations which do not disturb the Hamiltonian form of equations are referred to as "canonical" or "contact". Univalent canonical transformations (c = 1) are the ones most frequently employed. /37

If equations (4.10) are subjected to the canonical transformation

$$p_i = \frac{\partial V}{\partial q_i}, \quad \eta_i = -\frac{\partial V}{\partial \xi_i} \quad (i = 1, 2, \ldots, n), \tag{4.11}$$

where $V = V(t, q_1, \ldots, q_n, \xi_1, \ldots, \xi_n)$, then we arrive at a new system of Hamiltonian equations

$$\frac{d\xi_i}{dt} = \frac{\partial K}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{\partial K}{\partial \xi_i} \quad (i = 1, 2, \ldots, n) \tag{4.12}$$

with the Hamiltonian

$$K = H + \frac{\partial V}{\partial t}. \tag{4.13}$$

The theorem on canonical transformations suggests a means of integrating the equations of motion, and leads directly to the well-known Hamilton-Jacobi law.

Let us consider the Hamiltonian system of (4.10) and attempt to find such a canonical transformation which will result in the Hamiltonian equations in the following form:

$$\frac{d\xi_i}{dt} = 0, \quad \frac{d\eta_i}{dt} = 0 \quad (i = 1, 2, \ldots, n), \tag{4.14}$$

- that is, that the new characteristic function K will be identically equal to zero:

$$K = H + \frac{\partial V}{\partial t} = 0. \tag{4.15}$$

If we are able to find such a transformation, then the new system of equations arrived at (4.14) can be integrated directly, and its common integral will be written as follows:

$$\xi_i = \alpha_i, \qquad \eta_i = -\beta_i \qquad (i = 1, 2, \ldots, n), \tag{4.16}$$

where α_i and β_i are arbitrary constants.

The generating function of the transformation must satisfy equation (4.15), /38 which, on the basis of equation (4.11), can be written as follows:

$$\frac{\partial V}{\partial t} + H\left(t, q_1, \ldots, q_n, \frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_n}\right) = 0. \tag{4.17}$$

The equation thus obtained is called the Hamiltonian-Jacobi equation[1].

If the generating function V, which satisfies equation (4.17), contains n arbitrary constants, and the condition

$$\det\left(\frac{\partial^2 V}{\partial q_i \partial \xi_j}\right) \neq 0, \tag{4.18}$$

is met, then the function V will be the total integral of equation (4.17). By knowing the total integral of equation (4.17), it is then easy to obtain the total integral of the Hamiltonian equations of (4.10), by using formulas (4.16).

We thus come to still another theorem.

The Jacobi Theorem. Let the system of canonical Hamiltonian equations of (4.10) be given, and let V ($t, q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n$) be the total integral of the Hamilton-Jacobi equation (4.17); then, the total integral of system (4.10) can be represented as follows:

$$p_i = \frac{\partial V}{\partial q_i}, \quad \beta_i = \frac{\partial V}{\partial \alpha_i} \quad (i = 1, 2, \ldots, n), \tag{4.19}$$

[1] This equation is also called the Hamilton-Ostrogradskiy equation.

where α_i and β_i are arbitrary constants.

Corollary. If the mechanical system is conservative -- that is, if $\frac{\partial H}{\partial t} \equiv 0$ -- then the total integral of the Hamilton-Jacobi equation can be represented as follows:

$$V = -\alpha_1 t + W(q_1, q_2, \ldots, q_n), \tag{4.20}$$

where the constant α_1, in the case of scleronomous constraints and scleronomous transformation to generalized coordinates, denotes the total mechanical energy, while the function W satisfies the equation: /39

$$H\left(q_1, q_2, \ldots, q_n, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}, \ldots, \frac{\partial W}{\partial q_n}\right) = \alpha_1. \tag{4.21}$$

The Hamilton-Jacobi equation can be written in a somewhat different form if the system (4.10) is transformed beforehand, as follows:

$$\left.\begin{aligned} \xi_i &= q_i, \quad p_i = \eta_i \qquad (i = 1, 2, \ldots, m), \\ \xi_j &= p_j, \quad \eta_j = -q_j \quad (j = m+1, \ldots, n), \end{aligned}\right\} \tag{4.22}$$

this transformation is canonical, and leads to a system of Hamiltonian equations:

$$\frac{d\xi_i}{dt} = \frac{\partial H}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{\partial H}{\partial \xi_i} \quad (i = 1, 2, \ldots, n) \tag{4.23}$$

with the characteristic function:

$$H = H(t, \xi_1, \ldots, \xi_m, -\eta_{m+1}, \ldots, -\eta_n, \eta_1, \ldots, \eta_m, \xi_{m+1}, \ldots, \xi_n) \tag{4.24}$$

Then, instead of equation (4.17), we have

$$\frac{\partial V}{\partial t} + H\left(t, q_1, \ldots, q_m, -\frac{\partial V}{\partial p_{m+1}}, \ldots, -\frac{\partial V}{\partial p_n}, \frac{\partial V}{\partial q_1}, \ldots, \frac{\partial V}{\partial q_m}, p_{m+1}, \ldots, p_n\right) = 0. \tag{4.25}$$

In addition to the total integral of the Hamilton-Jacobi equation, the various partial integrals of this equation may also be useful. This problem has been studied by Lehmann-Filhes[1], whose results are stated below.

[1] See Lehmann Filhes, Astr. Nachr., Bd. 165, 1904.

If a partial integral of the Hamilton-Jacobi equation has been found, $V(t, q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_k)$, which depends upon $k < n$ arbitrary constants α_i, then it is possible to find 2k first integrals for the equations of motion

$$\frac{\partial V}{\partial \alpha_i} = \beta_i, \quad \frac{\partial V}{\partial q_i} = p_i \qquad (i = 1, 2, \ldots, k). \tag{4.26}$$

§ 5. Integration of the Hamilton-Jacobi Equation by the Separation of Variables

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The use of the method of separation of variables to integrate differential equations with partial derivatives of the second degree has been studied by V. G. Imshenetskiy [19]. However in the science of mechanics we are more interested in those cases of integrability of the Hamilton-Jacobi equation in which the metric of the space of generalized coordinates is delineated. Such cases of integrability have been studied by numerous writers (J. Liouville [20], P. Stackel [21], Morrera [22], Dall'Aqua [23], G. Pirro [24], Painleve [25], N. D. Moiseyev [26], etc.). Below we give two cases which generalize the results of [21] and [26] (See [27]). Here it should be noted that, as far as celestial mechanics and celestial ballistics are concerned, the cases dealt with by Liouville and Stackel, along with their generalizations, are the most useful ones[1]. The use of these theorems offers the possibility of a simpler and a more complete study of the nature of motion (See, for example, [28]).

Let us examine, now, a certain class of dynamic systems for which it is possible to demonstrate the existence of a partial integral of the Hamilton-Jacobi equation.

Theorem. If the kinetic energy of system T and its potential U are defined by the formulas

$$T = \frac{1}{2} b \sum_{i=1}^{k} a_i(q_i)\dot{q}_i^2 + T^*(\dot{q}_{k+1}, \ldots, \dot{q}_n, q_{k+1}, \ldots, q_n) \tag{5.1}$$

$$U = \frac{1}{b} \sum_{i=1}^{k} U_i(q_i) + U^*(q_{k+1}, \ldots, q_n), \quad k \leqslant n, \tag{5.2}$$

in which

$$b = \sum_{i=1}^{k} b_i(q_i), \tag{5.3}$$

where a_i, b_i, U_i, T*, U* are arbitrary functions of their arguments, then the Hamilton-Jacobi equation admits of the following integral:

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[1] See Yarov-Yarovoy, M. S. [29].

$$V = -ht + \sum_{i=1}^{k} \int \sqrt{2a_i(U_i - hb_i + \alpha_i)}\, dq_i + V^*, \tag{5.4}$$

where h and α_i are arbitrary constants.

Proof. In accordance with equation (2.1), we introduce the canonical moments:

$$p_i = \frac{\partial T}{\partial \dot{q}_i};$$

then, for the Hamiltonian function we will have

$$H = \frac{1}{2b} \sum_{i=1}^{k} \frac{p_i^2}{a_i} + T_2^*(p_{k+1}, \ldots, p_n, q_{k+1}, \ldots, q_n) -$$
$$- T_0^*(q_{k+1}, \ldots, q_n) - \frac{1}{b} \sum_{i=1}^{k} U_i(q_i) - U^*(q_{k+1}, \ldots, q_n).$$

According to equation (4.21), the Hamilton-Jacobi equation is written in the form

$$\sum_{i=1}^{k} \left[\frac{1}{2a_i} \left(\frac{\partial V}{\partial q_i} \right)^2 - U_i + hb_i \right] + b(T_2^* - T_0^* - U^*) + h \sum_{i=k+1}^{n} b_i(q_i) = 0. \tag{5.5}$$

We shall search for the integral of equation (5.5) in the form:

$$V = \sum_{i=1}^{k} V_i(q_i) + V^*(q_{k+1}, \ldots, q_n). \tag{5.6}$$

From equation (5.5) it is evident that the functions V_i must satisfy the equations

$$\left(\frac{dV_i}{dq_i} \right)^2 = 2a_i(U_i - hb_i + \alpha_i), \tag{5.7}$$

where α_i are arbitrary constants.

From this we find that

$$V_i(q_i) = \int \sqrt{2a_i(U_i - hb_i + \alpha_i)}\, dq_i \quad (i = 1, 2, \ldots, k). \tag{5.8}$$

Substituting in equation (5.6) the expressions for functions V_i obtained in equation (5.8) we arrive at /42

$$V = \sum_{i=1}^{k} \int \sqrt{2a_i(U_i - hb_i + \alpha_i)}\, dq_i + V^*, \tag{5.9}$$

where the function V* must satisfy the equation

$$T_2^*\left(\frac{\partial V^*}{\partial q_{k+1}}, \ldots, \frac{\partial V^*}{\partial q_n}, q_{k+1}, \ldots, q_n\right) - T_0^* - U^* + h\sum_{i=k+1}^{n} b_i(q_i) + \sum_{i=1}^{k} \alpha_i = 0. \tag{5.10}$$

From equations (4.26) and (5.9) it follows that the dynamic system under consideration admits of an integral of the Hamilton-Jacobi equation (5.4) containing k + 1 arbitrary constants.

Remark. Using the integral (5.4), it will be possible, on the basis of equation (4.26), to find k + 1 first integrals.

Corollary. If the kinetic energy and the potential are defined by the formulas

$$T = \frac{1}{2} b \sum_{i=1}^{n} a_i(q_i)\, \dot{q}_i^2, \tag{5.11}$$

$$U = \frac{1}{b} \sum_{i=1}^{n} U_i(q_i), \tag{5.12}$$

in which

$$b = \sum_{i=1}^{n} b_i(q_i),$$

then the Hamilton-Jacobi equation will admit of the total integral

$$V = -ht + \sum_{i=1}^{n} \int \sqrt{2a_i(U_i - hb_i + \alpha_i)}\, dq_i, \tag{5.13}$$

where the arbitrary constants α_i are associated by the relationship

$$\sum_{i=1}^{n} \alpha_i = 0. \tag{5.14}$$

This, in effect, is the Liouville theorem. Dynamic systems from which the conditions of equations (5.11) and (5.12) are met, are referred to as "Liouville systems". A study of Liouville systems may be found in an article by V.I. Arnol'd [30] (see also [31]). /43

The following theorem is a generalization of the work done by Stackel and N.D. Moiseyev (see [27]).

Theorem. Let there be n^2 arbitrary functions $\phi_{ij}(q_i)$ (i, j = 1, 2,..., n) which satisfy the condition

$$\Delta = \begin{vmatrix} \varphi_{11} & \varphi_{21} & \cdots & \varphi_{n1} \\ \varphi_{12} & \varphi_{22} & \cdots & \varphi_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{1n} & \varphi_{2n} & \cdots & \varphi_{nn} \end{vmatrix} \neq 0, \tag{5.15}$$

and n + 1 arbitrary functions $\Phi(q_1, q_2, \ldots, q_n)$ and U_i (q_i) $(i = 1, 2, \ldots, n)$. Then, if the kinetic energy T and the force function U are defined by the formulas

$$T = \frac{1}{2} \sum_{i=1}^{n} \left(\frac{a_i}{A_i} \dot{q}_i^2 + 2 \frac{\partial \Phi}{\partial q_i} \dot{q}_i + A_i b_i \right), \tag{5.16}$$

$$U = \sum_{i=1}^{n} A_i U_i, \tag{5.17}$$

where $A_i = \frac{1}{\Delta} \frac{\partial \Delta}{\partial \varphi_{i1}}$, and each of the coefficients a_i and b_i depends only upon the corresponding variable q_i, then the Hamilton-Jacobi equation has a total integral

$$V = \Phi + \sum_{i=1}^{n} \int \sqrt{a_i \left(b_i + 2U_i + 2\alpha_1 \varphi_{i1} + \sum_{j=2}^{n} \alpha_j \varphi_{ij} \right)} \, dq_i. \tag{5.18}$$

Proof. Inserting the moments p_i with the help of equation (2.1),

$$p_i = \frac{a_i}{A_i} \dot{q}_i + \frac{\partial \Phi}{\partial q_i},$$

we arrive at the following expression for the Hamiltonian H:

$$H = \frac{1}{2} \sum_{i=1}^{n} \left[\frac{A_i}{a_i} \left(p_i - \frac{\partial \Phi}{\partial q_i} \right)^2 - A_i b_i \right] - U. \tag{5.19}$$

Then, according to equation (4.21), the Hamilton-Jacobi equation can be written in the following form /44

$$\sum_{i=1}^{n} A_i \left[\frac{1}{2a_i} \left(\frac{\partial V}{\partial q_i} - \frac{\partial \Phi}{\partial q_i} \right)^2 - \frac{1}{2} b_i - U_i \right] = \alpha_1. \tag{5.20}$$

Making use of the identity

$$\sum_{i=1}^{n} \varphi_{i1} A_i = 1$$

instead of equation (5.20), we have

$$\sum_{i=1}^{n} A_i \left[\frac{1}{a_i} \left(\frac{\partial V}{\partial q_i} - \frac{\partial \Phi}{\partial q_i} \right)^2 - b_i - 2\alpha_1 \varphi_{i1} - 2U_i \right] = 0. \tag{5.21}$$

Equation (5.21) is satisfied if we stipulate that

$$\frac{\partial V}{\partial q_i} = \frac{\partial \Phi}{\partial q_i} + \sqrt{a_i \left(b_i + 2U_i + 2\alpha_1 \varphi_{i1} + \sum_{j=2}^{n} \alpha_j \varphi_{ij} \right)}, \tag{5.22}$$

where α_i are arbitrary constants.

Actually, substituting in equation (5.21) the values of $\frac{\partial V}{\partial q_i}$ from equation (5.22), we see that equation (5.21) is satisfied identically,

$$\sum_{i=1}^{n} A_i \sum_{j=2}^{n} \alpha_j \varphi_{ij} = \sum_{j=2}^{n} \sum_{i=1}^{n} \alpha_j A_i \varphi_{ij} = \frac{1}{\Delta} \sum_{j=2}^{n} \alpha_j \sum_{i=1}^{n} \varphi_{ij} \frac{\partial \Delta}{\partial \varphi_{i1}} \equiv 0,$$

since for j = 2, 3,..., n each of the quantities

$$\sum_{i=1}^{n} \varphi_{ij} \frac{\partial \Delta}{\partial \varphi_{i1}},$$

representing the sum of the products of the elements of the j-th line of the determinant by the cofactors of the elements of the first line, becomes zero. The condition of equation (4.18) is also satisfied.

Proceeding from the reasoning of Darboux [32] regarding the integration of the equations of motion of a dynamic system whose kinetic energy is a homogeneous function of generalized velocities, we can delineate still another case of integrability. /45

Theorem. Let there be $n(n + 2)$ continuous functions, each of which depends solely upon a single variable, namely

$$\varphi_{ij}(q_j) \qquad (i, j = 1, 2, \ldots, n),$$
$$U_i(q_i),\ a_i(q_i) \qquad (i = 1, 2, \ldots, n),$$

and let there be a differentiable function $\Phi(q_1, \ldots, q_n)$. We shall assume that

$$\Delta = \det |\varphi_{ij}(q_i)| \not\equiv 0.$$

If, now, the kinetic energy T and the potential U are defined by the formulas

$$T = \frac{1}{2} \sum_{i=1}^{n} \left[\frac{a_i}{A_i} \dot{q}_i^2 \cdot \sum_{j=1}^{n} A_j U_j + 2 \frac{\partial \Phi}{\partial q_i} \dot{q}_i \right], \tag{5.23}$$

$$U = \frac{\Delta}{\sum_{i=1}^{n} A_i U_i}, \tag{5.24}$$

in which

$$A_i = \frac{\partial \Delta}{\partial \varphi_{i1}} \qquad (i = 1, 2, \ldots, n), \tag{5.25}$$

then for the corresponding Hamilton-Jacobi equation the total integral will be:

$$V = \Phi + \sum_{i=1}^{n} \int \sqrt{a_i \left(2\alpha_i U_i + 2\varphi_{i1} + \sum_{j=2}^{n} \alpha_j \varphi_{ij}\right)}\, dq_i. \tag{5.26}$$

Proof. The Hamilton-Jacobi equation is written as follows

$$\frac{1}{\sum_{j=1}^{n} A_j U_j} \left\{ \sum_{i=1}^{n} \frac{A_i}{a_i} \left(\frac{\partial W}{\partial q_i} - \frac{\partial \Phi}{\partial q_i} \right)^2 - \Delta \right\} = \alpha_1, \tag{5.27}$$

where α_1 is the constant of integration.

Equation (5.27) can be written as follows /46

$$\sum_{i=1}^{n} A_i \left[\frac{1}{a_i} \left(\frac{\partial V}{\partial q_i} - \frac{\partial \Phi}{\partial q_i} \right)^2 - 2\varphi_{i1} - 2\alpha_1 U_i \right] = 0. \tag{5.28}$$

As can readily be verified, (5.26) really represents the total integral of the Hamilton-Jacobi equation (5.27).

§ 6. Integration of the Hamilton-Jacobi Equations in Spherical Coordinates

Integrating equations of motion by the Jacobi method depends to a great degree not only upon the structure of the potential, but also upon the space metric of the generalized coordinates. For example, the Liouville theorem, given in § 5, in the case of three degrees of freedom enables us to find the total integral of the Hamilton-Jacobi equation only with the use of isometric coordinates -- in particular, ellipsoidal coordinates.

Let us examine the motion of a point in a spherical system of coordinates within a force field with potential U (r, ϕ, λ). From equation (3.18) we find expression for the moments

$$p_r = \dot{r}, \quad p_\varphi = r^2 \dot{\varphi}, \quad p_\lambda = r^2 \dot{\lambda} \cos^2 \varphi.$$

The Hamiltonian is written as follows:

$$H = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\varphi^2 + \frac{1}{r^2\cos^2\varphi}p_\lambda^2\right) - U(r, \varphi, \lambda). \tag{6.1}$$

According to equation (4.21), integration of the equations of motion (3.19) is reduced to the problem of finding the total integral of the Hamilton-Jacobi equation

$$\left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V}{\partial \varphi}\right)^2 + \frac{1}{r^2\cos^2\varphi}\left(\frac{\partial V}{\partial \lambda}\right)^2 - 2U(r, \varphi, \lambda) = 2\alpha_1. \tag{6.2}$$

According to the first theorem of § 5, equation (6.2) has a partial integral, provided

$$U(r, \varphi, \lambda) = f(r) + \frac{\Phi(\varphi, \lambda)}{r^2}, \tag{6.3}$$

where f and Φ are arbitrary functions of their arguments. Substituting in (6.2) the expressions for U from (6.3), we obtain this Hamilton-Jacobi equation: /47

$$\left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V}{\partial \varphi}\right)^2 + \frac{1}{r^2\cos^2\varphi}\left(\frac{\partial V}{\partial \lambda}\right)^2 - 2f(r) - \frac{2\Phi(\varphi, \lambda)}{r^2} - 2\alpha_1 = 0. \tag{6.4}$$

Assuming

$$V(r, \varphi, \lambda) = V_1(r) + V_2(\varphi, \lambda) \tag{6.5}$$

and substituting this value in equation (6.4), we arrive at

$$\left(\frac{dV_1}{dr}\right)^2 - 2f(r) - 2\alpha_1 + \frac{1}{r^2}\left[\left(\frac{\partial V_2}{\partial \varphi}\right)^2 + \frac{1}{\cos^2\varphi}\left(\frac{\partial V_2}{\partial \lambda}\right)^2 + 2\Phi(\varphi, \lambda)\right] = 0. \tag{6.6}$$

We require that the function $V_2(\phi, \lambda)$ satisfy the equation

$$\left(\frac{\partial V_2}{\partial \varphi}\right)^2 + \frac{1}{\cos^2\varphi}\left(\frac{\partial V_2}{\partial \lambda}\right)^2 + 2\Phi(\varphi, \lambda) = \alpha_2, \tag{6.7}$$

in which α_2 is an arbitrary constant.

The function $V_1(r)$ is determined from the equation

$$\left(\frac{dV_1}{dr}\right)^2 - 2f(r) + \frac{\alpha_2}{r^2} - 2\alpha_1 = 0. \tag{6.8}$$

Separating the variables in this equation, and integrating, we arrive at the following integral:

$$V_1(r) = \int \sqrt{2f(r) + 2\alpha_1 - \frac{\alpha_2}{r^2}}\, dr. \tag{6.9}$$

On the basis of (4.19), we find the first integral of the equations of motion of (3.19), using this relationship:

$$V = -\alpha_1 t + \int \sqrt{2f(r) + 2\alpha_1 - \frac{\alpha_2}{r^2}}\, dr. \tag{6.10}$$

According to equation (4.26) we have

$$\int \frac{dr}{\sqrt{2f(r) + 2\alpha_1 - \frac{\alpha_2}{r^2}}} = t + \beta_1. \tag{6.11}$$

This integral enables us to study the planetocentric distance of a fairly remote satellite, in the case in which we may neglect the term with the spherical harmonic when analyzing the gravitational potential of the planet. /48

Remark. It is interesting to observe that for a field of forces of type (6.3), there is an analog to the Binet formula. Actually, if the potential of the field is defined by formula (6.3), then equations (3.19) assume the following form:

$$\left.\begin{aligned} \ddot{r} - r(\dot{\varphi}^2 + \dot{\lambda}^2 \cos^2\varphi) &= f_r' - \frac{2\Phi(\varphi, \lambda)}{r^3}, \\ \frac{d}{dt}(r^2\dot{\lambda}\cos^2\varphi) &= \frac{\Phi_\lambda'}{r^2}, \\ \frac{d}{dt}(r^2\dot{\varphi}) + r^2\dot{\lambda}^2 \sin\varphi\cos\varphi &= \frac{\Phi_\varphi'}{r^2}. \end{aligned}\right\} \tag{6.12}$$

In place of time t and radius-vector r, let us introduce the independent variable τ and the reciprocal of the distance u:

$$u = \frac{1}{r}, \quad d\tau = \frac{u^2}{c}\, dt, \tag{6.13}$$

where c is an arbitrary constant.

Then, system (6.12) assumes the following form

$$\left.\begin{aligned} u'' + (\lambda'^2 \cos^2\varphi + \varphi'^2) &= -\frac{c^2}{u^2} f'_r + 2\Phi u, \\ \frac{d}{d\tau}(\lambda' \cos^2\varphi) &= c^2 \Phi'_\lambda, \\ \varphi'' + \lambda'^2 \sin\varphi \cos\varphi &= c^2 \Phi'_\varphi, \end{aligned}\right\} \tag{6.14}$$

where the prime sign denotes differentiation with respect to τ.

Multiplying the latter two equations by λ' and ϕ', respectively, and integrating, we find the first integral:

$$\lambda'^2 \cos^2\varphi + \varphi'^2 = 2c^2(\Phi + c_1), \tag{6.15}$$

where c_1 is the constant of integration.

With the help of equation (6.15), the first of the equations of (6.14) is transformed as follows:

$$u'' + 2c_1 c^2 u = c^2 f'_u. \tag{6.16}$$

If in the potential (6.3) there is no second member ($\Phi \equiv 0$), then for $\phi = 0$, assuming that c_1 is equal to the area constants, we arrive at the Binet formula. Here the new independent variable is the longitude λ.

Formula (6.16) can be used in studying the motion of artificial earth satellites in a central gravitational field if we take advantage of the ideas of B. Garfinkel [33]. Equation (6.16), and also the Clairaut-Laplace equation [34], as well as the equations of A. I. Lur'ye [35] are quite convenient in the study of satellite motion provided the averaging method is used [36]. /49

If the potential has the form

$$U(r, \varphi) = f(r) + \frac{\Phi(\varphi)}{r^2}, \tag{6.17}$$

then the problem is reduced to one of quadrature.

Actually, since equation (6.7) does not contain an explicit factor of longitude λ, we can stipulate that

$$V_2 = \alpha_3 \lambda + \widetilde{V}_2(\varphi). \tag{6.18}$$

The substitution of (6.18) in equation (6.7) results in

$$\widetilde{V}'^2_2 + \frac{\alpha_3^2}{\cos^2\varphi} + 2\Phi(\varphi) - \alpha_2 = 0. \tag{6.19}$$

From the latter, with the help of quadratures, we arrive at

$$\tilde{V}_2(\varphi) = \int \sqrt{\alpha_2 - 2\Phi(\varphi) - \frac{\alpha_3^2}{\cos^2\varphi}}\, d\varphi. \tag{6.20}$$

Making due allowance for (6.5), (6.9) and (6.20), we obtain the total integral of the Hamilton-Jacobi equation

$$V = -\alpha_1 t + \int \sqrt{2f(r) - \frac{\alpha_3}{r^2} + 2\alpha_1}\, dr + \\ + \alpha_3\lambda + \int \sqrt{\alpha_2 - 2\Phi(\varphi) - \frac{\alpha_3^2}{\cos^2\varphi}}\, d\varphi. \tag{6.21}$$

We have used the given problem as an illustration of the results obtained by R. Barrer [37] and B. Garfinkel [33] in connection with the motion of artificial earth satellites.

§ 7. Motion Within a Central Field of Forces. The Two-Body Problem.

In celestial ballistics, in connection with the motion of a material point in a central field of forces, we are concerned not only with the two-body problem, but with the analysis of the motion of an artificial satellite of a spheriodal planet within the equatorial plane of the planet.

Therefore, let us examine the general problem of the motion of a material point in a central field of forces. /50

On the basis of equation (6.1), the Hamiltonian of the problem is:

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + \frac{p_\lambda^2}{r^2\cos^2\varphi}\right) - U(r). \tag{7.1}$$

The differential Hamiltonian-Jacobi equation

$$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial V}{\partial \varphi}\right)^2 + \frac{1}{r^2\cos^2\varphi}\left(\frac{\partial V}{\partial \lambda}\right)^2 - 2U(r) = 2\alpha_1 \tag{7.2}$$

is integrated by the separation of variables (see § 5).

If we designate

$$V = V_1(r) + V_2(\varphi) + V_3(\lambda) - \alpha_1 t, \tag{7.3}$$

then equation (7.2) breaks down into three differential equations:

$$V_1'^2 + \frac{\alpha_2^2}{r^2} - 2U(r) - 2\alpha_1 = 0, \tag{7.4}$$

$$V_2'^2 + \frac{\alpha_3^2}{\cos^2\varphi} - \alpha_2^2 = 0, \tag{7.5}$$

$$V_3' - \alpha_3 = 0, \tag{7.6}$$

in which α_i are arbitrary constants. These constants have a simple mechanical interpretation. The constant α_1 is the total mechanical energy; α_3 is the area constant which characterizes the amount of motion with respect to the z axis; and α_2 is the module of the kinetic moment with respect to the center of forces.

Integrating (7.4) - (7.6), and substituting the values of the function V_i that is found in formula (7.3), we obtain the following for the total integral of the Hamilton-Jacobi equation (7.2):

$$V = \int_{r_0}^{r} \sqrt{2U(r) - \frac{\alpha_2^2}{r^2} + 2\alpha_1}\, dr + \int_0^{\varphi} \sqrt{\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\varphi}}\, d\varphi + \alpha_3\lambda - \alpha_1 t, \tag{7.7}$$

where r_0 is a constant which will be selected later on[1]. /51

From (7.7), on the basis of Jacobi's theorem (see § 4), we find the first integrals of the problem:

$$\int_{r_0}^{r} \frac{dr}{\sqrt{2U(r) - \frac{\alpha_2^2}{r^2} + 2\alpha_1}} = t + \beta_1, \tag{7.8}$$

$$-\alpha_2 \int_{r_0}^{r} \frac{dr}{r^2\sqrt{2U(r) - \frac{\alpha_2^2}{r^2} + 2\alpha_1}} + \alpha_2 \int_0^{\varphi} \frac{d\varphi}{\sqrt{\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\varphi}}} = \beta_2, \tag{7.9}$$

$$\lambda - \alpha_3 \int_0^{\varphi} \frac{d\varphi}{\cos^2\varphi\sqrt{\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\varphi}}} = \beta_3. \tag{7.10}$$

Since the vector of the kinetic moment, on the basis of equations (7.5) and (7.6), maintains a constant direction, then the orbit of the point will be a plane curve, and the normal to the orbital plane will be co-linear with the vector of kinetic moment. It follows from equation (7.5) that motion is possible upon condition

[1] In formula (7.7) we take definite integrals so that there will be no unnecessary constants introduced into the expression for the total integral.

$$\alpha_2^2 - \alpha_3^2 \sec^2 \varphi \geqslant 0,$$

which can be written as follows

$$|\cos \varphi| \geqslant \left| \frac{\alpha_3}{\alpha_2} \right|. \tag{7.11}$$

If we use the symbol i to denote the inclination of the orbital plane to the basic coordinate plane, then, on the basis of equation (7.11), we have

$$\cos i = \frac{\alpha_3}{\alpha_2}. \tag{7.12}$$

According to equations (7.11) and (7.12), the spherical coordinate ϕ varies within the following limits

$$-i \leq \phi \leq +i.$$

The motion of a point is conveniently represented on a spherical surface (Figure 9) whose center lies at the center of forces. In our study of motion we shall make use of the following concepts. The line which marks the interception of the orbit with the basic coordinate plane we shall call the line of nodes. The points of intersection of this line with the celestial sphere we shall refer to the ascending and the descending nodes of the orbit; here the ascending node is the one such that, with transition through it, the point falls in the Northern Hemisphere (in Figure 9 the ascending angle is denoted by the symbol Ω, and the descending angle by the symbol υ). The position of the ascending node we shall define as the arc which is reckoned in the positive direction from the x axis on the basic coordinate plane. This arc we shall refer to as the "longitude of the ascending node". /52

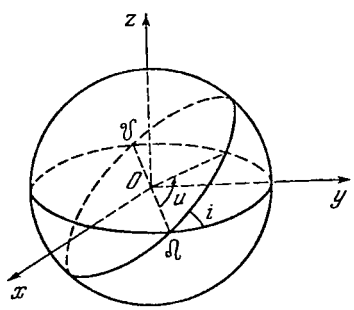

Figure 9

We shall begin by studying the general properties of motion making use of the integral of (7.8). We shall limit ourselves to the case in which the equation

$$2U(r) - \frac{\alpha_2^2}{r^2} + 2\alpha_1 = 0 \tag{7.13}$$

has two different positive roots

$$0 < r_p < r_a$$

Here, of course it is assumed that between roots r_p and r_a there are no other roots of equation (7.13). As is evident from (7.8), the radius-vector r always lies between the limits

$$r_p \leq r \leq r_a, \tag{7.14}$$

if it satisfies the above mentioned inequalities at the initial moment. Then r_p will be the least (pericentric) value of r, and r_a the greatest (apocentric) value of r. In place of r_p and r_a we can introduce the constants a and e, with the help of the following relations: /53

$$r_p = a(1-e), \qquad r_a = a(1+e). \tag{7.15}$$

The constant a we shall refer to as the "mean distance", the quantity e will be called the "measure of oblateness of the orbit" (quasi-eccentricity).

Let us assume that in the total integral (7.7), as well as in integrals (7.8) and (7.9), the lower limit of integration in the first term is r_p. Then (7.8) becomes

$$\int_{r_p}^{r} \frac{dr}{\sqrt{(r-r_p)(r_a-r)\Phi(r)}} = t + \beta_1, \tag{7.16}$$

where $\Phi(r)$ is defined by the equation

$$\Phi(r)(r-r_p)(r_a-r) = 2U(r) - \frac{\alpha_2^2}{r^2} + 2\alpha_1,$$

while, obviously $\Phi(r) \geq 0$ for all values of r which satisfy the conditions of (7.14).

In (7.16) we substitute

$$\beta_1 = -T. \tag{7.17}$$

If t = T, the equality $r = r_p$ is fulfilled: i.e., at moment T the moving point is at a minimal distance from the center of forces (at the pericenter). The quantity T we shall refer to as the moment of passage through the pericenter.

Replacing the variable in the integral (7.16)

$$r = a(1 - e\cos E), \tag{7.18}$$

we arrive at

$$\int_0^E \frac{dE}{\sqrt{\Phi}\,[a(1-e\cos E)]} = t - T, \tag{7.19}$$

from which it is evident that E is a regulating variable. Equation (7.19) can be referred to as the equation of time (in the two-body problem it is referred to as the Kepler equation).

If e = 0, then $\Phi(r) = \Phi(a)$ and $E = \sqrt{\Phi(a)}\,(t - T)$. With values of e which are absolutely small, and explicit expression for E in terms of time t can /54 be obtained with the help of the Lagrange formula [38], which is well-known in analysis.

After obtaining a value of the regulating variable E for a definite value of t from equation (7.19), with the help of equation (7.18) we proceed to find the value of the radius-vector r.

Now we return to formula (7.9), which defines the latitude, ϕ. Making use of (7.12) instead of (7.9), we obtain

$$\int_0^{\varphi} \frac{\cos\varphi\, d\varphi}{\sqrt{\sin^2 i - \sin^2\varphi}} - \beta_2 = \alpha_2 \int_{r_p}^{r} \frac{dr}{r^2\sqrt{2U(r) - \frac{\alpha_2^2}{r^2} + 2\alpha_1}}. \tag{7.20}$$

In place of ϕ we introduce the new variable u, which is the argument of latitude (Figure 9), being in the form of an angle reckoned in the orbital plane from the ascending node of the orbit:

$$\sin\varphi = \sin i \sin u. \tag{7.21}$$

From equation (7.20), with the help of (7.21), we arrive at

$$u - \beta_2 = \alpha_2 \int_{r_p}^{r} \frac{dr}{r^2\sqrt{2U(r) - \frac{\alpha_2^2}{r^2} + 2\alpha_1}}. \tag{7.22}$$

Let us now determine the meaning of β_2. Since with $r = r_p$, the equality $u = \beta_2$ is fulfilled, then β_2 is equal to the argument of latitude at the moment of passage through the pericenter. We now introduce the notation:

$$\beta_2 = \omega \tag{7.23}$$

and agree to call the quantity ω "the distance between the pericenter and the node". Formula (7.22) reduces to the following form:

$$u - \omega = \alpha_2 \int_{r_p}^{r} \frac{dr}{r^2 \sqrt{2U(r) - \frac{\alpha_2^2}{r^2} + 2\alpha_1}}. \tag{7.24}$$

Let us turn, now, to equation (7.10), which defines the longitude of the moving point. With the help of the substitution (7.21) we transform equation (7.10) to the following:

$$\lambda - \beta_3 = \text{arc tan (cos i tan u)}. \tag{7.25}$$

On the basis of equation (7.25), for u = 0, the equality $\lambda = \beta_3$ is fulfilled. /55
We now introduce the notation

$$\beta_3 = \Omega ; \tag{7.26}$$

then, instead of equation (7.25), we arrive at

$$\lambda - \Omega = \text{arctan (cos i tan u)}. \tag{7.27}$$

It is obvious that the quantity Ω is the longitude of the ascending node.

The Two-Body Problem. Let us apply the results thus far obtained to the two-body problem -- i.e., to the motion of two material points which are mutually gravitating according to Newtonian laws. If we limit ourselves to the study of the motion of one of the points (say the satellite) with respect to the system of coordinates with origin at the other point (the planet), and also restrict ourselves to constant directions of the axes, then the gravitation potential will be defined by this formula:

$$U(r) = \frac{f(M+m)}{r}, \tag{7.28}$$

where f is the constant of gravitation, M is the mass of the planet, m is the mass of the satellite, and r is the planetocentric distance of the satellite. Ordinarily, the mass of an artificial satellite as reflected in formula (M + m) (7.28) can be neglected.

For the case in which the mechanical energy is negative, from formulas (7.18), (7.19), (7.21), (7.24) and (7.27), we can obtain a solution to the problem by substituting the expression from equation (7.28) for the value of U:

$$r = a(1 - e\cos E), \tag{7.29}$$
$$E - e\sin E = M, \tag{7.30}$$

where

$$M = n(t - T),\ n^2 = \frac{fM}{a^3},$$
$$u - \omega = v, \tag{7.31}$$
$$\lambda - \Omega = \arctan(\cos i \tan u). \tag{7.32}$$

In formulas (7.29) - (7.30) we employ the following conventions:

$$\left.\begin{aligned} &\alpha_1 = -\frac{fM}{2a},\ \alpha_2 = \sqrt{fMp},\ \alpha_3 = \sqrt{fMp}\cos i,\\ &\beta_1 = -T,\ \beta_2 = \omega,\ \beta_3 = \Omega. \end{aligned}\right\} \tag{7.33}$$

Here a, e, i, Ω, ω, T are Keplerian elements: a is the measure semi-axis of the elliptical orbit; e is the eccentricity; i is the inclination; Ω is the longitude of the ascending node; T is the moment of passage through the pericenter; $p = a(1 - e^2)$ is the focal parameter; and M is the mean anomaly. /56

If we stipulate that

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}, \tag{7.34}$$

then, from equation (7.29) we have the following value for the radius-vector:

$$r = \frac{p}{1 + e\cos v}. \tag{7.35}$$

The quantity v, called the "true anomaly", is actually the polar angle reckoned in the orbital plane from the direction toward the pericenter.

§ 8. Integration of the Hamilton-Jacobi Equation in Spheroidal and in Paraboloidal Coordinates.

Let us consider the motion of a material point within a system of oblate spheroidal coordinates defined by the formulas of (3.20). We shall assume that the potential of the problem has this form:

$$U = U(r, \phi), \tag{8.1}$$

where r and ϕ are spherical coordinates (the potential does not depend upon the longitude of the moving point). In the spheroidal coordinates of (3.20), instead of (8.1) we have

$$U = U(\operatorname{sh}\psi, \sin\vartheta). \tag{8.2}$$

On the basis of (3.27) we have

$$L = \frac{c^2}{2}[J(\dot{\psi}^2 + \dot{\vartheta}^2) + \dot{\lambda}^2 \operatorname{ch}^2\psi \cos^2\vartheta] + U(\operatorname{sh}\psi, \sin\vartheta), \tag{8.3}$$

where

$$J = \operatorname{ch}^2\psi - \cos^2\vartheta. \tag{8.4}$$

Applying the second group of formulas of (2.1), we introduce the canonical moments

$$p_\psi = c^2 J\dot{\psi}, \quad p_\vartheta = c^2 J\dot{\vartheta}, \quad p_\lambda = c^2 \operatorname{ch}^2\psi \cos^2\vartheta \cdot \dot{\lambda}. \tag{8.5}$$

In these variables the Hamiltonian of the problem is written in the following form: /57

$$H = \frac{1}{2c^2}\left[\frac{1}{J}(p_\psi^2 + p_\vartheta^2) + \frac{1}{\operatorname{ch}^2\psi \cos^2\vartheta} p_\lambda^2\right] - U(\operatorname{sh}\psi, \sin\vartheta). \tag{8.6}$$

In accordance with (4.20), the problem of integrating equations of motion reduces to the construction of the total integral of the Hamilton-Jacobi equation:

$$\frac{1}{2c^2}\left\{\frac{1}{J}\left[\left(\frac{\partial V}{\partial\psi}\right)^2 + \left(\frac{\partial V}{\partial\vartheta}\right)^2\right] + \frac{1}{\operatorname{ch}^2\psi \cos^2\vartheta}\left(\frac{\partial V}{\partial\lambda}\right)^2\right\} - U = \alpha_1. \tag{8.7}$$

On the basis of the second theorem cited in § 5, we establish the most general form of the force function U, which will enable us to integrate the equations of motion of (3.28) in quadratures.

For application of this theorem we stipulate that

$$\left.\begin{aligned} &\varphi_{11} = c^2 \operatorname{th}^2\psi, \quad \varphi_{21} = c^2 \operatorname{tg}^2\vartheta, \quad \varphi_{31} = 0, \\ &\varphi_{12} = -\frac{1}{\operatorname{ch}^2\psi}, \quad \varphi_{22} = \frac{1}{\cos^2\vartheta}, \quad \varphi_{32} = 0, \\ &\varphi_{13} = \frac{1}{\operatorname{ch}^4\psi}, \quad \varphi_{23} = -\frac{1}{\cos^4\vartheta}, \quad \varphi_{33} = 1 \end{aligned}\right\} \tag{8.8}$$

and compile the determinant

$$\Delta = |\varphi_{ij}| = \frac{c^2(\operatorname{ch}^2\psi - \cos^2\vartheta)}{\cos^2\vartheta \operatorname{ch}^2\psi} = \frac{c^2 J}{\cos^2\vartheta \operatorname{ch}^2\psi}. \tag{8.9}$$

The coefficients A_i we determine with the help of the formula

$$A_i = \frac{1}{\Delta}\frac{\partial\Delta}{\partial\varphi_{i1}}.$$

It is not difficult to assure ourselves that

$$A_1 = \frac{\operatorname{ch}^2\psi}{c^2 J}, \quad A_2 = \frac{\cos^2\vartheta}{c^2 J}, \quad A_3 = \frac{1}{c^2 \operatorname{ch}^2\psi \cos^2\vartheta}. \tag{8.10}$$

For the quantities a_i we will have

$$a_1 = \operatorname{ch}^2\psi, \quad a_2 = \cos^2\vartheta, \quad a_3 = 1. \tag{8.11}$$

Comparing equations (8.3), (8.10) and (8.11), we see that the kinetic energy of a material point in oblate spheroidal coordinates actually has the form which is necessary for the second theorem of § 5. /58

To determine the total integral of equation (8.7), it is sufficient, as follows from (5.18) and (8.10) that the potential have this form:

$$U = \frac{\Phi_1(\psi) - \Phi_2(\vartheta)}{\operatorname{ch}^2\psi - \cos^2\vartheta}. \tag{8.12}$$

This form of the force function (8.12) is necessary for the integration of equation (8.7) by the method of separation of variables.

In accordance with (5.18) the total integral of equation (8.7) has the following form:

$$V = \sqrt{\alpha_3}\lambda + \int \sqrt{2c^2\Phi_1(\psi) + 2\alpha_1 c^2 \operatorname{sh}^2\psi - \alpha_2 + \frac{\alpha_3}{\operatorname{ch}^2\psi}}\, d\psi + \int \sqrt{-2c^2\Phi_2(\vartheta) + 2\alpha_1 c^2 \sin^2\vartheta + \alpha_2 - \frac{\alpha_3}{\cos^2\vartheta}}\, d\vartheta. \tag{8.13}$$

Remark. The force function (8.12), as a limiting case, contains within itself the potential of (6.17). In order to demonstrate this fact, we transform the quantities

$$\left.\begin{aligned} r_1 &= \sqrt{x^2 + y^2 + (z - ci)^2}, \\ r_2 &= \sqrt{x^2 + y^2 + (z + ci)^2}, \end{aligned}\right\} \tag{8.14}$$

to spheroidal coordinates ψ, ϑ:

$$\left.\begin{aligned} r_1 &= c(\operatorname{sh}\psi - i\sin\vartheta), \\ r_2 &= c(\operatorname{sh}\psi + i\sin\vartheta). \end{aligned}\right\} \tag{8.15}$$

We now introduce the spheroidal coordinates q_1 and q_2 with the help of the equalities

$$q_1 = c\operatorname{sh}\psi, \quad q_2 = \sin\vartheta. \tag{8.16}$$

Then, from (8.15) and (8.16) we obtain

$$q_1 = \frac{r_1 + r_2}{2}, \quad q_2 = \frac{r_2 - r_1}{2ci}. \tag{8.17}$$

Expanding the expressions of (8.14) in the Taylor series by powers of c, we obtain the following:

$$\left.\begin{aligned} r_1 &= r\left(1 - \frac{cz}{r^2}i + \ldots\right), \\ r_2 &= r\left(1 + \frac{cz}{r^2}i + \ldots\right). \end{aligned}\right\} \tag{8.18}$$

Substituting in (8.17), in place of r_1 and r_2, their expressions from (8.18), and taking the limit as $c \to 0$, we find that oblate spheroidal coordinates ultimately reduce to these spheroidal coordinates

$$\lim_{c\to 0} q_1 = r, \quad \lim_{c\to 0} q_2 = \sin\varphi. \tag{8.19}$$

Next, the most general form of the force function, which enables us to solve the problem in quadratures, is obtained from formula (8.12); in the latter formula, by virtue of (8.16), we substitute the following limiting values of (8.19) instead of ψ and ϑ:

$$U = \frac{\Phi_1(r) - \Phi_2(\varphi)}{r^2}, \tag{8.20}$$

this yields a potential like that of (6.17).

In problems of celestial ballistics the quantity c is actually quite small, and for this reason the spheroidal coordinates of a surface are close to spheroidal coordinates.

Now we shall study the motion of the material point in the paraboloidal coordinates of (3.39). We shall demonstrate, first, that the structure of the

kinetic energy in paraboloidal coordinates satisfies the condition of (5.2) of the second theorem of § 5.

With this purpose in mind we introduce the following system of functions:

$$\left.\begin{aligned} \varphi_{11} &= \xi, & \varphi_{21} &= \eta, & \varphi_{31} &= 0, \\ \varphi_{12} &= -\frac{1}{\xi}, & \varphi_{22} &= \frac{1}{\eta}, & \varphi_{32} &= 0, \\ \varphi_{13} &= -\xi, & \varphi_{23} &= -\eta, & \varphi_{33} &= 1. \end{aligned}\right\} \tag{8.21}$$

The determinant Δ of this system of functions ϕ_{ij} is as follows:

$$\Delta = \frac{\xi^2 + \eta^2}{\xi\eta}. \tag{8.22}$$

We also stipulate that

$$a_1 = \xi, \quad a_2 = \eta, \quad a_3 = \xi\eta. \tag{8.23}$$

From (8.22) and (8.23), we find that

$$A_1 = \frac{\xi}{\xi^2 + \eta^2}, \quad A_2 = \frac{\eta}{\xi^2 + \eta^2}, \quad A_3 = 1. \tag{8.24}$$

Substituting the expressions of (8.24) in formula (5.16), we obtain the kinetic energy corresponding to the Lagrangian of (3.41).

To discover the integrable cases we make use of formula (5.17). From (5.17) and (8.22) it follows that the Hamilton-Jacobi equation can be integrated by use of the method of separation of variables, provided the potential /60 has this form

$$U = \frac{\Phi_1(\xi) + \Phi_2(\eta)}{\xi^2 + \eta^2}, \tag{8.25}$$

where Φ_1 and Φ_2 are arbitrary functions of their own arguments.

We now introduce the canonical moments

$$p_\xi = (\xi^2 + \eta^2)\,\dot{\xi}, \quad p_\eta = (\xi^2 + \eta^2)\,\dot{\eta}, \quad p_\varphi = \xi\eta. \tag{8.26}$$

Then the Hamilton-Jacobi equation of (5.20) assumes this form:

$$\frac{1}{2}\left\{\frac{1}{\xi^2 + \eta^2}\left[\left(\frac{\partial V}{\partial \xi}\right)^2 + \left(\frac{\partial V}{\partial \eta}\right)^2\right] + \frac{1}{\xi\eta}\left(\frac{\partial V}{\partial \varphi}\right)^2\right\} - \frac{\Phi_1(\xi) + \Phi_2(\eta)}{\xi^2 + \eta^2} = h. \tag{8.27}$$

In accordance with (5.18) we then proceed to find the total integral of equation (8.27).

§ 9. Conditional-Periodic Functions

As will be demonstrated subsequently, motions describable by Hamiltonian systems which can be integrated by the Jacobi method with use of the separation of variables, possess the property of conditional periodicity. In a dynamic problem with two degrees of freedom, conditional periodic motion can be treated geometrically in the following manner.

Let the point in question move along the surface of a torus (Figure 10).

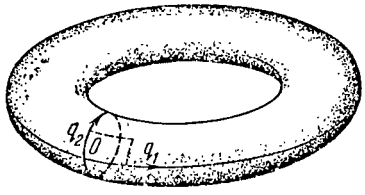

Figure 10.

As Lagrangian coordinates describing the motion of the point on the torus, we can take the longitude q_1, reckoned in the equatorial plane of the torus from a given direction assigned by point 0, and the latitude, q_2. The angular coordinates q_1 and q_2 will be expressed in radians. Let the coordinates of the point satisfy the equations

$$\frac{dq_1}{dt} = \omega_1, \quad \frac{dq_2}{dt} = \omega_2, \tag{9.1}$$

where the quantities ω_1 and ω_2 are called "frequencies". Then, if $\omega_1/\omega_2 = p/q$, /61 while p and q are whole numbers, after a certain interval of time $t = \frac{2\pi p}{\omega_1} = \frac{2\pi q}{\omega_2}$, the point will have returned to its initial position, having made q revolutions along the meridian, and p revolutions along the parallel. The motion of the point will be periodic; however, the trajectory of the point will be closed following several "revolutions".

If, however, the ratio p/q is an irrational number (the frequencies ω_1 and ω_2 are incommensurable), then the point will never return to its initial position. It is this sort of motion which is called conditional-periodic in celestial mechanics. The trajectory described by the point can be plotted on a plane in Cartesian coordinates. Since, according to (9.1), q_1 and q_2 are linear functions of time, then motion on the plane (q_1, q_2) is represented by a straight line.

So-called conditional periodic functions are used to describe conditional periodic forms of motion. If, on the torus, we assign the function $f(q_1, q_2)$, which is expanded in the Fourier series

$$f(q_1, q_2) = \sum_{p,\, q=-\infty}^{\infty} a_{pq} \exp[i(pq_1 + qq_2)],$$

then, in correspondence with (9.1), the variation of this function with respect to time will be described by this formula

$$f(t) = f[q_1(t),\ q_2(t)] = \sum_{p,\ q=-\infty}^{\infty} a_{pq} \exp[i(p\omega_1 + q\omega_2)t + \alpha_{pq}]. \qquad (9.2)$$

Functions like (9.2) are encountered in the solution of most problems met with in celestial mechanics and celestial ballistics, or in the study of perturbed motion. These functions are called "conditional-periodic".

If the mechanical system possesses n degrees of freedom, then for a geometrical interpretation of the motion one can adopt an n-dimensional torus. The position of the depicted system of the point is defined by n conditional coordinates $x_1(t),\ldots,x_n(t)$.

Definition. The function $f(x_1,x_2,\ldots,\ x_n,\ y_1,\ y_2,\ldots,\ y_n)$ is referred to as a conditional-periodic with respect to the arguments $x_1,\ x_2,\ldots,\ x_n$ which have a period $\omega = \{\omega_1,\omega_2,\ldots,\ \omega_n\}$, provided the following identity obtains: /62

$$f(x_1+\omega_1,\ x_2+\omega_2,\ \ldots,\ x_n+\omega_n,\ y_1,\ y_2,\ \ldots,\ y_n) \equiv \equiv f(x_1,\ x_2,\ \ldots,\ x_n,\ y_1,\ y_2,\ \ldots,\ y_n). \qquad (9.3)$$

Here the quantities $x_1,\ x_2,\ldots,\ x_n$ can be treated by angular coordinates on an n-dimensional torus, or else as orthogonal coordinates of a point in n-dimensional space. Then the period ω must be regarded as an n-dimensional vector. The components of the vector ω_i we shall call "elementary periods".

It is not difficult to demonstrate the justfication of the following theorems.

Theorem 1. If the function $f(x_1,\ x_2,\ldots,\ x_n,\ y_1,\ y_2,\ldots,\ y_n)$ possesses the period ω, then it will also possess the period $\lambda\omega = \{\lambda\omega_1,\ \lambda\omega_2,\ldots,\ \lambda\omega_n\}$, where λ is any whole number.

Theorem 2. If the function $f(x_1,\ x_2,\ldots,\ x_n,\ y_1,\ y_2,\ldots,\ y_n)$ is conditional-periodic with respect to $x_1,\ x_2,\ldots,\ x_n$, and if it has two periods

$$\omega_1 = \{\omega_{11},\ \omega_{12},\ \ldots,\ \omega_{1n}\},\ \ \omega_2 = \{\omega_{21},\ \omega_{22},\ \ldots,\ \omega_{2n}\},$$

then their vector sum

$$\omega_1 + \omega_2 = \{\omega_{11} + \omega_{21},\ \omega_{12} + \omega_{22},\ \ldots,\ \omega_{1n} + \omega_{2n}\}$$

is also a period.

Corollary. If the function (f x_1, x_2,..., x_n, y_1, y_2,..., y_n) has k periods

$$\omega_i=\{\omega_{i1}, \omega_{i2},..., \omega_{in}\} \qquad (i=1, 2,..., k),$$

then any linear combination of the vectors ω_i with interval coefficients is also a period:

$$\sum_{i=1}^{k} \lambda_i\omega_i = \left\{\sum_{i=1}^{k} \lambda_i\omega_{i1}, \sum_{i=1}^{k} \lambda_i\omega_{i2}, ..., \sum_{i=1}^{k} \lambda_i\omega_{in}\right\}. \tag{9.4}$$

We shall assume that no function possesses an infinitely small period: in other words, the absolute value of a vector ω cannot be less than any preassigned number however small. If this condition is met, then we can accept the following theorem:

Theorem 3. If the vectors ω and $\lambda\omega$ are periods of the function $f(x_1, x_2,..., x_n, y_1, y_2,..., y_n)$ then λ is a rational number.

Proof. We shall make use of a theorem from the theory of numbers. If λ is an arbitrary irrational number, then it is always possible to designate two whole numbers p and q such that the quantity $|p + q\lambda|$ will be less than any preassigned number. /63

Let us assume the contrary, namely that λ is an irrational number. Then we can always designate two whole numbers p and q such that the quantity $(p + q\lambda)\omega$, which by virtue of Theorem 1 is a period of function f, will be arbitrarily small. But if that were so, f would have an infinitely small period, which is impossible.

Corollary. Every period for any given direction can be represented as a multiple of the smallest period for that particular direction.

Theorem 4. If the function f (x_1, x_2,..., x_n, y_1, y_2,..., y_n) is conditional-periodic with respect to x_1, x_2,..., x_n, then there exists a system of periods ω_1, ω_2,..., ω_k such that any of the period of function f are representable in the form

$$\omega = \sum_{i=1}^{k} \lambda_i\omega_i, \tag{9.5}$$

where λ_i represents whole numbers, and where the number of periods k does not exceed the number of variables n.

Proof. Let us limit ourselves to the case n = 2. Let $f(x_1, x_2, y_1, \ldots, y_n)$ be periodic conditional with respect to x_1 and x_2. Let us consider all possible periods of this function, disposed in the order of their increasing absolute values:

$$|\omega^{(1)}| \leqslant |\omega^{(2)}| \leqslant |\omega^{(3)}| \leqslant \cdots$$

This will leave us with an infinite sequence, since by virtue of Theorem 3, the set of periods is denumerable, while the case of a function with an infinitely small period has already been excluded.

Let us select from this sequence the period $\omega_1 = \omega^{(1)}$, and the first period following it, $\omega_2 = \omega^{(2)}$, which does not coincide in direction with its predecessor. On the plane (x_1, x_2) let us construct a network of parallelograms whose sides are defined by the vectors ω_1 and ω_2 (Figure 11).

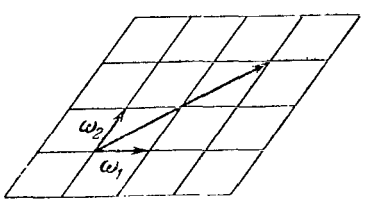

Figure 11.

Let us regard the period which is colinear with ω_1. Let us assume the contrary -- namely that we have found a period ω which is characterized by a vector whose end does not coincide with the node of the constructed network of parallelograms. Then, according to the corollary of Theorem 3, either the period ω_1 will be a multiple of ω, or the period ω_1 is a multiple of ω. /64

But the first is impossible by assumption, and the second is impossible by virtue of the choice of the period ω_1.

In the general case, one can conclude that to any period of arbitrary direction there must correspond a vector whose end coincides with the node of the network of parallelograms; for in the opposite event, this vector would have to be smaller in absolute value than $|\omega_2|$, which contradicts the choice of the vector ω_2.

Definition. The aggregate of periods which satisfy the conditions of Theorem 4 is called a "periodic system".

If we consider the conditional-periodic function $f(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$ with n periods, then in an n-dimensional space there will be a lattice (the analog of the network of parallelograms) which is defined by a periodic system. The periodic system can be selected in various ways. The periodic system $\{\omega_1, \omega_2, \ldots, \omega_n\}$ can be replaced with another having the same number of periods

$$\omega_i' = \sum_{k=1}^{n} \lambda_{ik}\omega_k \quad (i = 1, 2, \ldots, n).$$

If the determinant $\det |\lambda_{ik}|$ is equal to ± 1, then the periodic system is called "simple". With the help of transition from the variables x_i to the new variables w_i, the function f can be transformed in such a way that it will have the simple periodic system

$$\begin{array}{l} (1, 0, \ldots, 0), \\ (0, 1, \ldots, 0), \\ \ldots\ldots\ldots\ldots \\ (0, 0, \ldots, 1). \end{array}$$

§ 10. Conditional-Periodic Motions

/65

Suppose that for the Hamilton-Jacobi equation (4.17) we have found its total integral in the following form by the method of separation of variables:

$$V = -\alpha_1 t + \sum_{i=1}^{n} V_i(q_i, \alpha_1, \alpha_2, \ldots, \alpha_n). \tag{10.1}$$

Then the generalized moments p_i will be functions only of the corresponding conjugate coordinates

$$p_i = V_i'(q_i, \alpha_1, \alpha_2, \ldots, \alpha_n). \tag{10.2}$$

Since in practical cases the moments p_i in a Hamiltonian occur only in the second degree, then as a result of the separation of the variables in the Hamilton-Jacobi equation, and integrating the function V_i', we will have the following structure (CM. §5):

$$V_i'^2 = \Phi_i(q_i). \tag{10.3}$$

If the relations of (10.3) hold, then we may encounter three types of variation in the coordinates q_i: libration, rotation and asymptotic motion (see [28]).

Let q_i' and q_i'' be two adjacent routes of the equation

$$\Phi_i(q_i) = 0, \tag{10.4}$$

and let the quantity $\Phi_i(q_i)$ assume positive values for all values of q_i which

satisfy the condition

$$q_i' < q_i < q_i'',$$

In such a case the moments will approach zero only when $q_i = q_i'$ and $q_i = q_i''$. If q_i' and q_i'' are simple routes, then q_i will assume all the numerical values included between q_i' and q_i''. To express this situation, we say that libration occurs only with respect to the coordinate q_i, while the quantities q_i' and q_i'' are the "limits of libration". For libration on the plane (q_i, p_i) there exists the closed curve $p_i^2 - \Phi_i(q_i) = 0$. If, with change in the constance of integration, the closed curves retract toward a certain point, the latter is called the center of libration. It is to this center of libration that the stationary solution corresponds. /66

If q_i' and q_i'' are simple routes of equation (10.4), between which, in the case of variation in the numerical values of integration no new routes arise, then the state of equilibrium is stable. On the other hand, if with variation in α_i between q_i' and q_i'' some new routes appear, then the condition of equilibrium may be unstable. In such a case, for a certain system of values of the constant α_i, one of the routes q_i' and q_i'' becomes multiple, and then, for certain initial conditions of the coordinate, q_i will approach the state of equilibrium asymptotically. This is referred to as "limiting motion".

If the function $\Phi_i(q_i)$ is periodic with respect to q_i (without limitation of generality we can assume that the period is 2π) then the motion is referred to as rotational[1]. (In rotational motion of the coordinate, q_i is a monotonically increasing function of time). This type of motion occurs if $\Phi_i(q_i + 2\pi) \equiv \Phi_i(q_i)$, and if $\Phi_i(q_i) > 0$.

Let us assume now that for every coordinate the motion is either rotational or librational, and then proceed to examine the integral

$$J_i = \oint p_i dq_i \quad (i = 1, 2, \ldots, n) \tag{10.5}$$

(this integral, in the case of vibration, is taken over the closed loop defined by the equation $p_i^2 - \Phi_i(q_i) = 0$, and, in the case of rotation, between the limits of 0 and 2π). In place of the canonical elements α_i we introduce the

[1] By the introduction of the rotational coordinate $Q_i = \sin q_i$, rotational motion reduces to librational motion. However, if we do this, the libration limits become independent of the initial conditions.

new variables J_i. Then V can be expressed as a function of J_i and q_i. The generalized coordinate q_i will be replaced with the quantity w_i, which is defined by the relationship

$$w_i = \frac{\partial V}{\partial J_i} \qquad (i = 1, 2, \ldots, n). \tag{10.6}$$

The quantities J_i and w_i will now become new conjugate canonical elements. They are referred to as "action-angle elements" (J_i represents the variables of "action", w_i the variables of "angle" type). These so-called action angle elements were introduced by Delon. Later on they were put to use by Bohr and Sommerfeld in quantum physics. /67

Let us examine certain of the properties of the canonical elements which have been introduced. To do so, we calculate the increment of w_i for a full cycle of variation in the coordinate q_k:

$$\Delta_k w_i = \oint \frac{\partial w_i}{\partial q_k} dq_k. \tag{10.7}$$

From (10.6) and (10.1) we learn that

$$\frac{\partial w_i}{\partial q_k} = \frac{\partial^2 V}{\partial J_i \partial q_k} = \sum_{l=1}^{n} \frac{\partial^2 V_l}{\partial J_i \partial q_k} = \frac{\partial}{\partial J_i} \sum_{l=1}^{n} \frac{\partial V_l}{\partial q_k} = \frac{\partial}{\partial J_i} \left(\frac{\partial V_i}{\partial q_k} \right).$$

Substituting the expression obtained in (10.7), we then have

$$\Delta_k w_i = \frac{\partial}{\partial J_i} \oint \frac{\partial V_i}{\partial q_k} dq_k = \frac{\partial J_k}{\partial J_i} = \delta_{ik}, \tag{10.8}$$

in which δ_{ik} is a Kronecker symbol.

Thus, if any coordinate q_k prevents the full cycle of variation, then the corresponding variable w_k is varied by a whole unit (it should be noted that physically such a variation is not always possible, and that therefore the result arrived at here characterizes a purely mathematical aspect of angular variables).

The reverse proposition is also true. If the angular variable w_k increases by a whole unit, then the coordinate q_k, which is uniquely defined in terms of w_k, in the case of libration returns to its initial value, and in the case of rotation increases to 2π.

Let us examine the structure of the solution defined by the total integral of equation (10.1). If this integral is regarded as an arbitrary function of the canonical transformation

$$\left.\begin{aligned} p_i &= \frac{\partial V}{\partial q_i}, \\ \varphi_i &= \frac{\partial V}{\partial \alpha_i} \end{aligned}\right\} \qquad (i = 1, 2, \ldots, k) \tag{10.9}$$

of variables p_i, q_i to variables α_i, ϕ_i, then, on the basis of equations (4.11)-(4.13), the new Hamiltonian K will be dependent only upon the quantity α_i: /68

$$K = K(\alpha_1, \alpha_2, \ldots, \alpha_n), \tag{10.10}$$

in other words, the variables ϕ_i are cyclic, and the general solution of the problem can be written as follows:

$$\alpha_i = \text{const}, \quad \varphi_i = \omega_i t + \beta_i, \quad \omega_i = \frac{\partial K}{\partial \alpha_i}. \tag{10.11}$$

In practical problems, the new Hamiltonian K by virtue of equation (10.1) usually depends only upon the constant of the generalized integral of energy α_1, and hence

$$\omega_1 = \frac{\partial K}{\partial \alpha_1} = 1, \quad \omega_2 = \ldots = \omega_n = 0, \tag{10.12}$$

so that the solution of (10.11) becomes

$$\alpha_i = \text{const}, \quad \varphi_1 = t + \beta_1, \quad \varphi_2 = \beta_2, \quad \ldots, \quad \varphi_n = \beta_n. \tag{10.13}$$

On the basis of equation (10.8), the motion is conditional-periodic. This means that for its description it is most convenient to use canonical variables such that the periodic system will be simple. Let us perform the canonical transformation to variables α_i^*, ϕ_i^* with the help of the generating function

$$V^* = \sum_{k=1}^{n} f_k(\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)\varphi_k + g(\alpha_1^*, \ldots, \alpha_n^*), \tag{10.14}$$

which yields the following:

$$\left.\begin{aligned} \alpha_i &= f_i(\alpha_1^*, \ldots, \alpha_n^*), \\ \varphi_i^* &= \sum_{l=1}^{n} \frac{\partial f_l}{\partial \alpha_i^*} \varphi_l + \frac{\partial g}{\partial \alpha_i^*}. \end{aligned}\right\} \tag{10.15}$$

The general solution of the corresponding system of equations of motion may have the form of equation (10.11), but in that case K will no longer depend merely upon α_i^*, and, consequently, not only ω_i^* but also other values of $\omega_i^* = K'_{\alpha_i}$ will in general be different from zero. Just as in § 9 we can /69
assume that the motion takes place along n-dimensional torii $\alpha_i^* = \text{const}$, while ϕ_i^* will play the role of angular variables defining the position of the point on a torus. For an arbitrary choice of the variables α_i^* and ϕ_i^*, a periodic system is not simple. Obviously, a system of canonical "action-angle" variables, defined by formulas (10.5)-(10.6), can be obtained from transformations like that of (10.15). The elementary periods for these will be equal either to zero or to unity. This peculiarity of a "angle-action" variables makes them particularly convenient in practical situations.

Therefore, by virtue of equation (10.11), we have

$$w_i = \nu_i t + \delta_i, \quad \nu_i = \frac{\partial V}{\partial J_i}, \tag{10.16}$$

where J_i, δ_i are constants. According to (10.8), the old variables q_i are conditional periodic functions of the angular variables w_i. This means that the coordinates $q_l (l = 1, \ldots, n)$ can be expanded in multiple Fourier series:

$$q_l = \sum_{i_1, \ldots, i_n = -\infty}^{+\infty} c^{(l)}_{(i_1, i_2, \ldots, i_n)} e^{2\pi\sqrt{-1}[(i_1\nu_1 + \cdots + i_n\nu_n)t + (i_1\delta_1 + \cdots + i_n\delta_n)]}. \tag{10.17}$$

Or, making use of the vector form of recording, and substituting

$$\left.\begin{aligned} \mathbf{i} &= \{i_1, i_2, \ldots, i_n\}, \quad \boldsymbol{\delta} = \{\delta_1, \delta_2, \ldots, \delta_n\}, \\ \boldsymbol{\nu} &= \{\nu_1, \nu_2, \ldots, \nu_n\}, \\ (\mathbf{i}, \boldsymbol{\nu}) &= i_1\nu_1 + i_2\nu_2 + \cdots + i_n\nu_n, \\ (\mathbf{i}, \boldsymbol{\delta}) &= i_1\delta_1 + i_2\delta_2 + \ldots + i_n\delta_n, \end{aligned}\right\} \tag{10.18}$$

in place of (10.17) we will then have

$$q_l = \sum_{(i)} c_l^{(i)} e^{2\pi\sqrt{-1}[(\mathbf{i}, \boldsymbol{\nu})t + (\mathbf{i}, \boldsymbol{\delta})]} \quad (l = 1, 2, \ldots, n). \tag{10.19}$$

In the theory of perturbed motion the properties of the vector ν turn out to be quite substantial. If between the frequencies ν_i there does not exist the identical relationship

$$(\lambda, \nu) = 0, \tag{10.20}$$

where λ is a vector with interval components, then the dynamic system is referred to as "non-degenerate". If the relationship (10.20) is fulfilled identically, then eigendegeneracy takes place, and the system is referred to as "eigendegenerate". If the relationship $(\lambda, \nu) = 0$ is fulfilled only when certain initial conditions are given, we say that "random degeneracy" takes place. Finally, there is still another possibility in connection with degeneracy. If the lower and the upper limits of libration for k coordinates coincide, then in an n-dimensional space of generalized coordinates, the trajectory fills a region whose number of measurements is n - k. In this event we speak of "limiting degeneracy". /70

Let us take an example of limiting degeneracy. We shall assume that a point is moving on a rotating ellipse. Then, during the process of motion, the point will everywhere densely fill the annulus whose outer and inner radii are defined by the pericenter and the apocenter. With respect to the radius-vector the motion will be of vibrational character. If, however, we alter the initial conditions in such a way that the eccentricity of the ellipse approaches zero, then the limits of libration will approach each other, and the annulus will contract into a circle. In the limiting case, instead of a two-dimensional region we obtain a one-dimensional region. With zero value of eccentricity, limiting degeneracy appears.

Now let us establish one of the basic properties of conditional-periodic motion [39].

Theorem. If the motion is of librational character and is, moreover, non-degenerate, then the trajectory will everywhere densely fill the region of generalized coordinates which is limited by the hypersurfaces defined by the limits of libration.

Proof. For simplicity we limit ourselves to a system with three degrees of freedom. We shall take "angle-action" variables as our canonical variables. Then the system will be simple-periodic, and will be equal to unity. Let us introduce a rectangular system of coordinates (Figure 12) with origin at a certain point 0 in a space of angular variables w_1, w_2, w_3. Let e_1, e_2, e_3, be the unit vectors all the corresponding coordinate axes. Then, on the basis of (10.16), the trajectory defined by the equations

$$w_k = \nu_k t + \delta_k \quad (k = 1, 2, 3)$$

is a straight line, while the direction cosines of this straight line are proportional to the frequencies ν_1, ν_2, ν_3. By the condition of the theorem, /71

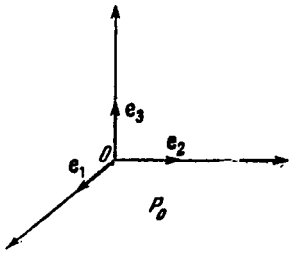

Figure 12.

the motion is non-degenerative, and, consequently the values of ν_k will be incommensurable.

In this proof we make use of the concept of the equivalence of the two points $(w_1, 2_2, w_3)$ and (w_1', w_2', w_3'). These two points are described as "equivalent" when the vector joining them is of the form

$$\sum_{k=1}^{3} \lambda_k \nu_k e_k,$$

where λ_k represents whole numbers.

Having replaced every point with one equivalent to it, we can then limit ourselves to an examination of the behavior of the trajectory within a unit cube. Let P_0, P_1, P_2,... represent the points of intersection of the trajectory with the faces of the cube. For convenience we shall assume that the origin of coordinates 0 coincides with the point P_0. As a consequence of the incommensurability of the frequencies ν_k, no one of the points P_i will coincide with any other.

On each of these faces we can construct an infinite set of vectors $\overrightarrow{P_m P_n}$, which together join P_i points. It is obvious that it is always possible to detect at least one vector $\overrightarrow{P_m P_n}$ whose absolute value is less than that of any preassigned number however small. This follows from the theorem takes from the theory of numbers which was cited in § 9.

Now let us consider the distribution of points P_i on any of the faces of the cube -- for example, on face e_2, e_3. By reason of the non-degeneracy, there will be found a terminal number σ such that the point P_σ will lie on the face e_2, e_3. Taking this point as the initial one, let us examine all possible vectors $\overrightarrow{P_\sigma P_n}$. These vectors define all points of the trajectory lying on face e_2, e_3. We shall show that these points cannot lie on a straight line.

If we assume the contrary, then for any two points P_1' and P_2' of this face of the cube which belong to the trajectory, we have /72

$$P_i'\left\{\frac{\nu_1}{\nu_3}x_i - \mathrm{E}\left(\frac{\nu_1}{\nu_3}x_i\right);\quad \frac{\nu_2}{\nu_3}x_i - \mathrm{E}\left(\frac{\nu_2}{\nu_3}x_i\right)\right\}\quad (i=1,2)$$

and, in addition we have the point

$$P_\sigma\left\{\frac{\nu_1}{\nu_3}-E\left(\frac{\nu_1}{\nu_3}\right);\ \frac{\nu_2}{\nu_3}-E\left(\frac{\nu_2}{\nu_3}\right)\right\}.$$

By reason of the coplanarity of the vectors $\vec{P}_\sigma P_i'$, we obtain

$$\begin{vmatrix} \frac{\nu_1}{\nu_3}-E\left(\frac{\nu_1}{\nu_3}\right), & \frac{\nu_2}{\nu_3}-E\left(\frac{\nu_2}{\nu_3}\right), & 1 \\ \frac{\nu_1 x_1}{\nu_3}-E\left(\frac{\nu_1 x_1}{\nu_3}\right), & \frac{\nu_2 x_1}{\nu_3}-E\left(\frac{\nu_2 x_1}{\nu_3}\right), & 1 \\ \frac{\nu_1 x_2}{\nu_3}-E\left(\frac{\nu_1 x_2}{\nu_3}\right), & \frac{\nu_2 x_2}{\nu_3}-E\left(\frac{\nu_2 x_2}{\nu_3}\right), & 1 \end{vmatrix}=0. \tag{10.21}$$

Multiplying the elements of the first row, first by x_1, and then by x_2, deducting from the results obtained, the elements of the second and the third rows, respectively, we transform equations (10.21) into the following:

$$\begin{vmatrix} \frac{\nu_1}{\nu_3}-E\left(\frac{\nu_1}{\nu_3}\right), & \frac{\nu_2}{\nu_3}-E\left(\frac{\nu_2}{\nu_3}\right), & 1 \\ E\left(\frac{\nu_1 x_1}{\nu_3}\right)-x_1 E\left(\frac{\nu_1}{\nu_3}\right), & E\left(\frac{\nu_2 x_1}{\nu_3}\right)-x_1 E\left(\frac{\nu_2}{\nu_3}\right), & x_1-1 \\ E\left(\frac{\nu_1 x_2}{\nu_3}\right)-x_2 E\left(\frac{\nu_1}{\nu_3}\right), & E\left(\frac{\nu_2 x_2}{\nu_3}\right)-x_2 E\left(\frac{\nu_2}{\nu_3}\right), & x_2-1 \end{vmatrix}=0.$$

Since, by the condition of the theorem ν_i are incommensurable, then

$$\frac{\nu_i}{\nu_3}-E\left(\frac{\nu_i}{\nu_3}\right)\neq 0 \quad (i=1,2),$$

and therefore the preceding condition is rewritten as follows:

$$\begin{vmatrix} E\left(\frac{\nu_1 x_1}{\nu_3}\right)-x_1 E\left(\frac{\nu_1}{\nu_3}\right), & E\left(\frac{\nu_2 x_1}{\nu_3}\right)-x_1 E\left(\frac{\nu_2}{\nu_3}\right) \\ E\left(\frac{\nu_1 x_2}{\nu_3}\right)-x_2 E\left(\frac{\nu_1}{\nu_3}\right), & E\left(\frac{\nu_2 x_2}{\lambda_3}\right)-x_2 E\left(\frac{\nu_2}{\nu_3}\right) \end{vmatrix}=0.$$

Dividing the elements of the first row by $x_1 - 1$, and converting to the limit for $x_1 \to \infty$, we arrive at /73

$$\begin{vmatrix} \frac{\nu_2}{\nu_3}-E\left(\frac{\nu_2}{\nu_3}\right), & 1 \\ E\left(\frac{x_2\nu_2}{\nu_3}\right)-x_2, & x_2 E\left(\frac{\nu_2}{\nu_3}\right)-1 \end{vmatrix}=0.$$

But this is inconsistent with the condition of the incommensurability of the frequencies ν_2 and ν_3. Consequently, the set of points $\{P_i\}$ belonging to the face e_2, e_3 cannot lie on a straight line. Analogous reasoning can be applied for the other faces.

It is now clearly evident that the points of the trajectory which lie on the faces of the cube fill the latter densely everywhere. Consequently, the trajectories pass as close to any of the points of the cube as one cares to stipulate.

J. Vinti, while studying the motion of artificial earth satellites on the basis of the problem integrable with the help of Stackel's theorem [40], took up the question of mean motions within Stackel's systems [41]. Below we extend the results obtained by Vinti to arbitrary dynamic systems which are integrable with the help of Jacobi's theorem by the separation of variables.

Theorem. If the dynamic system is such that a single-valued Hamiltonian exists within the considered region of variation of the generalized coordinates, and if each of the generalized coordinates possesses a libration variation, and is, moreover, a single-valued and differentiable function of canonical variables of the "angle" type, then the mean frequency for each of the coordinates is equal to the frequency of the corresponding angular variable.

Proof. Let T be the interval of time containing a whole number of cycles N_k of variation of the coordinate q_k. By the term "mean frequency of n_k" we shall understand the quantity

$$n_k = \lim_{T\to\infty} \frac{N_k}{T}, \tag{10.22}$$

assuming that this limit exists. Now it is necessary to show that for conditional-periodic motions

$$n_k = \nu_k, \tag{10.23}$$

where, according to (10.16) /74

$$\nu_k = \frac{\partial V}{\partial J_k}.$$

If all values of ν_i are commensurable, then we can find a positive value ν_0 and also certain natural numbers $m_1, m_2, \ldots, m_n$ such that

$$\nu_k = m_k \nu_0 \quad (k = 1, 2, \ldots, n). \tag{10.24}$$

By reason of (10.8) and (10.24), for real motion we have

$$w_k(t) = w_k(0) + m_k \nu_0 t, \tag{10.25}$$

and for the interval of time $T = 1/\nu_0$, the variation in w_k, according to equation (10.25), will be

$$\Delta w_k = m_k. \tag{10.26}$$

In this case each of the coordinates q_k completes m_k full cycles, and the motion is periodic, with the period of $1/\nu_0$. Thus, the mean frequency n_k is equal to

$$n_k = \frac{m_k}{T} = m_k \nu_0 = \nu_k. \tag{10.27}$$

If the frequencies ν_k are incommensurable, then we stipulate

$$\xi_k = \nu_k : \nu_1, \tag{10.28}$$

in which case at least one of the quantities ξ_k must be irrational. As above, we find

$$w_k(t) = w_k(0) + \xi_k \nu_1 t. \tag{10.29}$$

Making use of Dirichlet's theorem to the effect that if among the real numbers $\xi_1, \xi_2, \ldots, \xi_n$, then there is at least one which is irrational, and the system of inequalities

$$\left|\xi_k - \frac{m_k}{P}\right| < P^{-1-\frac{1}{n}} \quad (k = 1, 2, \ldots, n) \tag{10.30}$$

admits of an infinite set of integral solutions with respect to P and n, and a solution for P is not limited above.

Let us consider those values of $T = P/\nu_1$ for which there exists a whole number which satisfies the inequalities of (10.30). In the course of this interval of time, each of the angular coordinates w_k varies from the initial value $w_k(0)$ to the magnitude /75

$$w_k(T) = w_k(0) + P\xi_k. \tag{10.31}$$

But, on the basis of (10.30)

$$P\xi_k = m_k + \eta_k, \tag{10.32}$$

while

$$|\eta_k| < P^{-\frac{1}{n}}, \tag{10.33}$$

so that

$$w_k(T) = w_k(0) + m_k + \eta_k. \tag{10.34}$$

While P assumes increasing values which satisfy the conditions of (10.30), each of the magnitudes η_k, in correspondence with (10.33), approaches zero. On the basis of the foregoing theorem, and also equation (10.34), there exists an infinite set of values of T for which the trajectory in a space of canonical variables of "angle" type approaches as close to the points as one desires; here, $\Delta w_k = m_k$ (m_k is a whole number).

Let

$$q_k(0) = f_k(w_1(0), w_2(0), \ldots, w_k(0)), \tag{10.35}$$

and for t = T

$$q_k(T) = f_k[w_1(0) + m_1 + \eta_1, \ldots, w_k(0) + m_k + \eta_k]. \tag{10.36}$$

If, according to (10.30), P increases without limit, then q_k approaches the quantities

$$q_k = f_k[w_1(0) + m_1, \ldots, w_k(0) + m_k]. \tag{10.37}$$

It is evident from equation (10.37) that $\Delta w_k = m_k$ (k = 1, 2,..., n). But in this event the time interval T contains an integral number of full cycles of variation in each of the coordinates. According to the definition of equation (10.22), we arrive at the following mean frequency:

$$n_k = \lim_{T\to\infty} \frac{m_k}{T}. \tag{10.38}$$

This limit exists, so that /76

$$\frac{m_k}{T} = \frac{\nu_1 m_\nu}{P},$$

and therefore

$$\lim_{T\to\infty} \frac{m_k}{P} = \xi_k. \tag{10.39}$$

Thus, $n_k = \nu_1 \xi_k = \nu_k$. The theorem is proved.

§ 11. The Problem of Perturbed Motion

Both the problems of classical celestial mechanics, and the new problems of celestial ballistics, in the majority of cases cannot be resolved in quadratures in finite form. Therefore, celestial mechanics makes extensive use of approximation methods for solving systems of differential equations of motion, particularly the various forms of the method of successive approximations.

Let us suppose that we are studying a mechanical system whose Hamiltonian function H, on the basis of physical or mathematical considerations, can be broken down into two portions:

$$H = H_0 + R, \tag{11.1}$$

then the function R in the region of variation of canonical variables which interests us remains quite small in absolute value in comparison with the first term H_0. Usually, the function R contains one or several small parameters, and drops out when these parameters approach zero. The role of the small parameter can be played by the mass of the planet in the sun-planet-spaceship or by a quantity proportional to the flattening of the planet, in the problem of the motion of an artificial satellite of the planet. Sometimes the small parameter is introduced artificially, for example, by means of transformation of variables.

If within the region of phase space which interests us p_i, q_i, the following condition is always met

$$|R| \ll |H_0|, \tag{11.2}$$

then we can always assume that the Hamiltonian of the problem has this form:

$$H = H_0(p, q) + R(\mu, p, q, t), \tag{11.3}$$

in which

$$R(0, p, q, t) \equiv 0. \tag{11.4}$$

The term H_0 is called "unperturbed" and is constructed usually in such a way that the simplified problem with a Hamiltonian H_0 either can be integrated in quadratures, or admits of a certain family of solutions. The motion defined by the Hamiltonian H_0 is called "unperturbed", while the motion defined by the

Hamiltonian function H, is called "perturbed". The function R is called perturbed or "perturbation".

Let us write the canonical equations of perturbed motion:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}. \tag{11.5}$$

If $\mu = 0$, then instead of (11.5) we have these equations:

$$\frac{dq_i}{dt} = \frac{\partial H_0}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H_0}{\partial q_i}. \tag{11.6}$$

We shall assume that system (11.6) can be integrated by the Jacobi method. Let

$$V = V(t, \alpha, q) \tag{11.7}$$

be the total integral of the Hamilton-Jacobi equation describing unperturbed motion.

Taking (11.7) as the generating function, we transform system (11.5) to new variables α_i and β_i. In correspondence with (4.11) - (4.13), the new system of canonical equations has this form:

$$\frac{d\alpha_i}{dt} = \frac{\partial R}{\partial \beta_i}, \quad \frac{d\beta_i}{dt} = -\frac{\partial R}{\partial \alpha_i}. \tag{11.8}$$

The new variables α_i and β_i are referred to as osculating elements in celestial mechanics.

System (11.8) is solved by the method of successive approximations; as zero approximation for the variables α_i and β_i we take the constant values corresponding to unperturbed motion: /78

$$\alpha_i = \alpha_i^{(0)}, \quad \beta_i = \beta_i^{(0)}. \tag{11.9}$$

The method of successive approximations enables us to represent the sought-for quantities in the form of series arranged in order of increasing degrees of the small parameter:

$$\alpha_i = \alpha_i^{(0)} + \sum_{k=1}^{\infty} \mu^k \alpha_i^{(k)}, \quad \beta_i = \beta_i^{(0)} + \sum_{k=1}^{\infty} \mu^k \beta_i^{(k)}. \tag{11.10}$$

The individual terms of these series are called perturbations or inequalities, and the exponent of the small parameter in the term under consideration is referred to as the order of perturbation.

Depending upon their analytical structure, perturbations are reduced to three categories:

1) a perturbed type

$$At^k, \tag{11.11}$$

where A is a certain constant coefficient which depends upon the initial values of the elements, and k is a natural number; these are called "circular";

2) a perturbed type

$$A \sin (\nu t + \delta), \tag{11.12}$$

where A, ν, δ are constant quantities which depend upon the initial conditions; these are referred to as "periodic";

3) perturbations having the form

$$At^k \sin (\nu t + \delta),\ k \neq 0, \tag{11.13}$$

are called "mixed".

The delineation of perturbations of these three types is conditioned by the character of the solution of the unperturbed problem in the case of libration motion. Since, according to (10.19), generalized coordinates in unperturbed motion are represented in the form of multiple Fourier series, then the perturbation function in the system of equations of perturbed motion (11.9) is also expanded in multiple Fourier series, this system being independent of the variable in which time appears. From this it is obvious that with integration of system (11.9) by the method of successive approximations, the series which represent perturbed motion may contain only terms of the type (11.11) - (11.13). /79

An extremely important question in both celestial mechanics and celestial ballistics is that of the combined effect of all the sought-for and mixed perturbations. Can we expect that, as a result of circular perturbations, the perturbed and the unperturbed values of the osculating elements will eventually diverge by an indefinitely wide margin? Or is it possible that the combination of all circular and mixed perturbations is actually a periodic function of time? Is it not possible, as well, that circular perturbations are really defects produced by the mathematical methods employed?

In order to discover the essence of this problem let us consider the

differential equation

$$\frac{dx}{dt} - \cos \mu t = 0, \tag{11.14}$$

in which μ is the small parameter. If we look for a solution to this equation in the form of series of degrees of a small parameter, then we get

$$x = x_0 + t - \ldots \tag{11.15}$$

At the same time, in finite form the general solution of this equation is written as follows:

$$x = x_0 + \frac{1}{\mu} \sin \mu t. \tag{11.16}$$

As is apparent from (11.15), perturbations of any order are circular, while from (11.16) it follows that in combination they yield a periodic perturbation with period $2\pi : \mu$. Since the quantity μ is small, then the period of perturbed variation in the quantity x will be very large. An infinite combination of circular perturbations x corresponds to a single-long period inequality. Thus, in the example considered, circular perturbations have arisen as a result of a weakness inherent in the method of successive approximations.

We should ask ourselves whether this same thing might not take place in problems of the motion of celestial bodies. Many studies have been devoted to this problem, most recently those made by V. I. Arnol'd [42], Yu. Mozer [43], and others, in which significant progress was made. In Chapter 6 we shall examine this question in further detail.

CHAPTER II

NEWTONIAN GRAVITATIONAL POTENTIAL OF AN ABSOLUTE SOLID

§ 1. The Potential of Volume Masses. Laplace's Equation. /80

There are many situations in classical celestial mechanics in which it is quite satisfactory to replace real gravitating masses with material points. This is due to the fact that the dimensions of gravitated bodies frequency are neglibly small in comparison with the distances which separate them. In the case of celestial ballistics, however, the situation is entirely different. Either during the course of the entire flight, or during individual stages of motion, the spaceship is in the immediate vicinity of the gravitating body, from which it is separated by a distance which is of the same order as the dimensions of the body itself. In this situation it is necessary to take very careful account of the perturbing influence exerted by the shape of the gravitating body. Therefore, the methods used to compile the gravitational potential of the planet are of great interest from the purely practical point of view.

In studying the potentials of celestial bodies we shall proceed from two assumptions.

1) The gravitating body is an absolute solid and is undeformed.

2) The gravitating body is fairly similar to a sphere with radial distribution of density.

We shall adopt a rectangular Cartesian system of coordinates with origin at a certain point 0 of the body, the coordinate axes being directed along the main axes of the ellipsoid of inertia of the body (Figure 13). Let κ (ξ, η, ζ) be the density of the body at the moving point M (ξ, η, ζ) and let $\Delta = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$ be the system between the point M and the point P (x, y, z), which is outside the body.

Then the gravitational potential[1] of the element of mass dm which includes /81 point M is defined by the equation

$$dU = \frac{f\,dm}{\Delta} = \frac{f\kappa\,d\tau}{\Delta}, \tag{1.1}$$

where f is the gravitational constant, $d\tau$ is the element of volume. Integrating with respect to volume occupied by the body we find an expression for the

[1] We should remember that the term "potential" in the theory of gravitation was introduced by Gauss [44].

gravitational potential of the entire body at the external point

$$U(x, y, z) = f\iiint_{\tau} \frac{\varkappa\, d\tau}{\Delta}. \qquad (1.2)$$

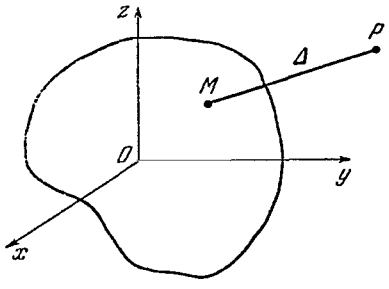

Figure 13.

In formula (1.2) the integration is performed over the entire volume occupied by the body under consideration. The potential U is the differentiated function of coordinates through the entire external space.

The gravitational force F exerted by the body is described by the formula

$$F = \text{grad } U. \qquad (1.3)$$

performing the two partial differentiations on equation (1.2) with respect to x, y, z, we establish that within the external space the potential U satisfies Laplaces equation:

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0. \qquad (1.4)$$

If we assume that the density $\varkappa(\xi, \eta, \zeta)$ has continuous first-order partial derivatives within the body, then it can be shown that at internal points of the body the potential satisfies the Poisson equation: /82

$$\Delta U = -4\pi\varkappa(x, y, z). \qquad (1.5)$$

The potential of the body can be calculated directly with use of formula (1.2). This direct method, however, is not the best one. Since the potential U in external space satisfies Laplace's equation, and, therefore is a harmonic function, then it is frequently preferable to employ a different method for finding the potential -- a method associated with the equations (1.4)-(1.5). When this is done it is necessary to solve Dirichlet's external problem. We shall proceed on the basis of Dirichlet's theorem, the proof of which may be found by the reader in various special textbooks on the theory of the Newtonian potential [45, 46]. This theorem is as follows.

Theorem. If the density $\varkappa$ within the body possesses continuous first-order partial derivatives, then the function U (x, y, z), which is regular at infinity, satisfies the following equation:

$$\Delta U = \begin{cases} 0 & \text{outside the body,} \\ -4\pi\varkappa & \text{inside the body,} \end{cases} \qquad (1.6)$$

and coincides with the gravitational potential of the body.

Let us note that from the condition of regularity of the potential at infinity, the following condition results:

$$\lim_{\rho \to \infty} \rho U = fm, \tag{1.7}$$

where $\rho = \sqrt{x^2 + y^2 + z^2}$, while $m = \iiint \varkappa \, d\tau$ is the mass of the gravitating body.

The effectiveness of determining the gravitational potential by the second method depends upon the choice of coordinates. Frequently it is more convenient to approach the solution in curvilinear coordinates q_1, q_2, q_3, rather than in rectangular coordinates. When this is done, the coordinates must be so choosen that the Laplace equation will be soluble by the separation of variables. First of all we transform Laplace's equation to new variables q_1, q_2, q_3 associated with rectangular coordinates: /83

$$\left.\begin{aligned} x &= x(q_1, q_2, q_3), \\ y &= y(q_1, q_2, q_3), \\ z &= z(q_1, q_2, q_3). \end{aligned}\right\} \tag{1.8}$$

The simplest form of Laplace's equation is obtained orthogonal coordinates (see (3.4), Chapter I).

As a result of calculations, we obtain the following for the Laplacian:

$$\Delta U = \frac{1}{H_1 H_2 H_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{H_2 H_3}{H_1}\frac{\partial U}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{H_1 H_3}{H_2}\frac{\partial U}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{H_1 H_2}{H_3}\frac{\partial U}{\partial q_3}\right)\right\}, \tag{1.9}$$

where the quantities H_i are the Lamé coefficients, defined by formulas (3.6), Chapter I.

Spherical and degenerate ellipsoidal coordinates are most frequently employed in problems of celestial ballistics. Laplace's equation admits of separation of variables in both of these systems of coordinates.

Making use of formulas (3.16) in Chapter I for the Lamé coefficients in a system of spherical coordinates, we can use equation (1.9) to obtain Laplace's equation in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{r^2 \cos\varphi}\frac{\partial}{\partial \varphi}\left(\cos\varphi \frac{\partial U}{\partial \varphi}\right) + \frac{1}{r^2 \cos^2\varphi}\frac{\partial^2 U}{\partial \lambda^2} = 0. \tag{1.10}$$

In the new studies made on the theory of the motion of artificial earth satellites by J. P. Vinti [40], M. D. Kislik [47] and others, the potential of terrestrial gravitation is found from Laplace's equation written in one of the systems of degenerate ellipsoidal coordinates. We make use of the spheroidal coordinates (3.20) Chapter I, analogous to the Thiele coordinates:

$$x = c \operatorname{ch}\psi \cos\vartheta \cos\lambda,$$
$$y = c \operatorname{ch}\psi \cos\vartheta \sin\lambda,$$
$$z = c \operatorname{sh}\psi \sin\vartheta.$$

From (1.9), with the help of expressions for the Lamé coefficients (3.25) Chapter I, we arrive at the following form of Laplace's equation in oblate spheroidal coordinates: /84

$$\frac{1}{c^2(\operatorname{ch}^2\psi - \cos^2\vartheta)}\left\{\frac{1}{\cos\vartheta}\frac{\partial}{\partial\vartheta}\left(\cos\vartheta\frac{\partial U}{\partial\vartheta}\right) + \frac{1}{\operatorname{ch}\psi}\frac{\partial}{\partial\psi}\left(\operatorname{ch}\psi\frac{\partial U}{\partial\psi}\right) + \left(\frac{1}{\cos^2\vartheta} - \frac{1}{\operatorname{ch}^2\psi}\right)\frac{\partial^2 U}{\partial\lambda^2}\right\} = 0. \quad (1.11)$$

The same method can be used to obtain Laplace's equation in prolate spheroidal coordinates, as defined by the transformation formulas (3.29) Chapter I:

$$\frac{1}{c^2(\operatorname{ch}^2 v - \cos^2 u)}\left\{\frac{1}{\sin u}\frac{\partial}{\partial u}\left(\sin u\frac{\partial U}{\partial u}\right) + \frac{1}{\operatorname{sh} v}\frac{\partial}{\partial v}\left(\operatorname{sh} v\frac{\partial U}{\partial v}\right) + \left(\frac{1}{\sin^2 u} + \frac{1}{\operatorname{sh}^2 v}\right)\frac{\partial^2 U}{\partial\omega^2} = 0. \quad (1.12)$$

In solving problems in the theory of the potential, it is convenient to make use of the expansion of harmonic functions in series in terms of spherical functions. We shall give some brief information on these spherical functions in the following section.

§ 2. Spherical Functions. Lagrangian Polynomials

Let us consider Laplace's equation in spherical coordinates (1.10), and look for partial solutions of this equation in the following form:

$$U(r, \varphi, \lambda) = R(r)\Phi(\varphi)\Lambda(\lambda). \quad (2.1)$$

Obviously, determining any of the multipliers of equation (2.1), as a result of the separation of variables in equation (1.10), reduces to the integration of an ordinary differential equation of second degree.

If we multiply equation (1.10) by $r^2 \cos^2\phi$, and transfer the term which depends upon λ to the right, we arrive at the following equation, following substitution in Laplace's equation:

$$\cos^2\varphi\left[\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Phi\cos\varphi}\frac{d}{d\varphi}\left(\cos\varphi\frac{d\Phi}{d\varphi}\right)\right] = -\frac{1}{\Lambda}\frac{d^2\Lambda}{d\lambda^2}.$$

This is possible, however, only if both members of the equation are constant quantities. Designating this constant with the symbol k^2, we obtain the following equations for determining R, Φ and Λ: /85

$$\frac{d^2\Lambda}{d\lambda^2} + k^2\Lambda = 0, \tag{2.2}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{k^2}{\cos^2\varphi} - \frac{1}{\Phi\cos\varphi}\frac{d}{d\varphi}\left(\cos\varphi\cdot\frac{d\Phi}{d\varphi}\right). \tag{2.3}$$

On the basis of the same reasoning, both members of equation (2.3) are equal to one and the same constant, which for convenience we shall designate with the symbol n (n + 1). Then we have the following:

$$\frac{1}{\cos\varphi}\frac{d}{d\varphi}\left(\cos\varphi\cdot\frac{d\Phi}{d\varphi}\right) + \left[n(n+1) - \frac{k^2}{\cos^2\varphi}\right]\Phi = 0, \tag{2.4}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - n(n+1)R = 0. \tag{2.5}$$

Thus we see that determining any of the multipliers leads to integration of equations (2.2), (2.4) and (2.5). Here the choice of the constants k and n must be such that any of the solutions of equation (2.1) arrived at will be a harmonic function. Then the solution of boundary-value problems of the Newtonian potential can be obtained by means of the superposition of various partial solutions, as is usually done when the Fourier method for integrating equations in partial derivatives is used.

The general solution of equation (2.2) has this form

$$\Lambda = A\cos k\lambda + B\sin k\lambda,$$

where A and B are constants of integration.

As regards equation (2.4), in the theory of the potential we make use only of those solutions which correspond to the natural values of k and n, with the proviso that k must not exceed n. Partial solutions of equation (2.4) depend upon the constants k and n, and therefore it is convenient to use the designation $P_n^k(\sin\phi)$.

Combinations of solutions of equations (2.2) and (2.4) of the form

$$Y_n(\varphi, \lambda) = \sum_{k=0}^{n} P_n^k(\sin\varphi)(A_{nk}\cos k\lambda + B_{nk}\sin k\lambda), \tag{2.6}$$

where A_{nk} and B_{nk} are arbitrary constants, are called "general spherical functions of the n-th degree[1].

[1] Spherical functions were first introduced by Laplace.

In studying the solutions of equation (2.4), we begin with the case k = 0. /86
Substituting

$$y = \Phi(\varphi), \quad x = \sin\varphi,$$

we first transform this equation to the form

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \left[n(n+1) - \frac{k^2}{1-x^2}\right]y = 0. \tag{2.7}$$

With k = 0, instead of (2.7) we will have

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0. \tag{2.8}$$

Partial solutions of this equation $P_n^0(x)$ will subsequently be denoted by the symbol $P_n(x)$[1].

It is not difficult to demonstrate that the polynomial

$$z = (x^2 - 1)^n$$

satisfies the following differential equation:

$$(x^2-1)\frac{dz}{dx} - 2nxz = 0.$$

If we differentiate this equation with respect to x (n + 1), then we arrive at the following equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{dz^{(n)}}{dx}\right] + n(n+1)z^{(n)} = 0. \tag{2.9}$$

Substituting $y = z^{(n)}$ in place of (2.9), we arrive at equation (2.8). From this it is possible to conclude that the function

$$y = \frac{d^n}{dx^n}(x^2-1)^n \tag{2.10}$$

[1] Equation (2.8) is a partial case of Gauss' equation (hypergeometric equation). The property of the integrals of this equation is most fully and rigidly studied in the analytical theory of differential equations. A detailed presentation of the theory of spherical functions can be found in the monographs by E. Hobson [48] and J. Lense [49].

is a partial solution of the equation which interests us. Since equation (2.8) is linear and homogeneous, then the product of function (2.10) by any constant will also be a solution of this equation.

In the theory of the potential we shall make use of solutions of equation (2.8) arrived at by the Rodrigues formula, which is itself derived from equation (2.10): /87

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n} \tag{2.11}$$

The Rodrigues formula defines a polynomial of the n-th degree with respect to x; this polynomial is called a Legendre polynomial.

The first of these polynomials which may be encountered in expansions of the gravitational potential of the earth, and which are employed in the theory of motion of artificial earth satellites, are as follows (Figure 14):

$$\left.\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2}, \\
P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x, \\
P_4(x) &= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}, \\
P_5(x) &= \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x, \\
P_6(x) &= \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16}, \\
P_7(x) &= \frac{429}{16}x^7 - \frac{693}{16}x^5 + \frac{315}{16}x^3 - \frac{35}{16}x, \\
P_8(x) &= \frac{6435}{128}x^8 - \frac{12012}{128}x^6 + \frac{6930}{128}x^4 - \frac{1260}{128}x^2 + \frac{35}{128}, \\
P_9(x) &= \frac{12155}{128}x^9 - \frac{25740}{128}x^7 + \frac{18018}{128}x^5 - \frac{4620}{128}x^3 + \frac{315}{128}x.
\end{aligned}\right\} \tag{2.12}$$

As is apparent from the Rodrigues formula, the Legendre polynomial is an analytical function, and therefore we can make use of a Cauchy integral:

$$f^{(n)}(x) = \frac{n!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-x)^{n+1}}, \tag{2.13}$$

where C is an arbitrary Jordan curve, and represent the Legendre polynomial with the formula /88

$$P_n(x) = \frac{1}{2^n \cdot 2\pi i} \oint_C \frac{(z^2-1)^n}{(z-x)^{n+1}}\,dz, \tag{2.14}$$

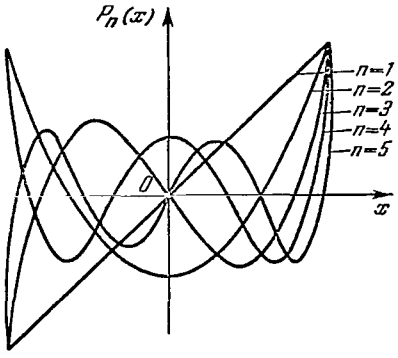

Figure 14.

known as the Schläfli formula.

We transform the integral of (2.14) to a new variable α, defined by the relationship

$$\alpha = 2\frac{z-x}{z^2-1}.$$

Following simple transformations, instead of equation (2.14) we arrive at

$$P_n(x) = \frac{1}{2\pi i}\oint_C \frac{d\alpha}{\alpha^{n+1}\sqrt{1-2\alpha x+\alpha^2}}. \qquad (2.15)$$

Comparing equations (2.15) and (2.13), we establish the fact that

$$P_n(x) = \frac{1}{n!}\left[\frac{d^n}{d\alpha^n}\left(\frac{1}{\sqrt{1-2\alpha x+\alpha^2}}\right)\right]_{\alpha=0},$$

in other words, the Legendre polynomial $P_n(x)$ is equal to the coefficient when α^n in the Maclaurian series

$$\frac{1}{\sqrt{1-2\alpha x+\alpha^2}} = \sum_{n=0}^{\infty}\alpha^n P_n(x). \qquad (2.16)$$

The left-hand member of this equation is referred to as a generating function. /89

We now establish certain important properties of the Legendre polynomial. Differentiating equation (2.16) with respect to α, we find that

$$(x-\alpha)(1-2\alpha x+\alpha^2)^{-\frac{3}{2}} = \sum_{n=1}^{\infty} n\alpha^{n-1}P_n(x),$$

or

$$(x-\alpha)\sum_{n=0}^{\infty}\alpha^n P_n(x) = (1-2\alpha x+\alpha^2)\sum_{n=1}^{\infty} n\alpha^{n-1}P_n(x).$$

Equating the coefficients which have identical degrees of α in the right and in the left members, we arrive at the following recurrent relationship:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0. \qquad (2.17)$$

But without going into details of the calculations, we can state the following property of Legendre polynomials:

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{2}{2n+1} & \text{if } m = n. \end{cases} \tag{2.18}$$

This means that the system of Legendre polynomials is orthogonal and that it may be used for the expansion of functions in series on the basis of Legendre polynomials.

Let us return now to equation (2.8) and study the general case for $k \neq 0$. We shall show that the partial solution of this equation is the function

$$P_n^k(x) = (1-x^2)^{\frac{k}{2}} \frac{d^k P_n(x)}{dx^k}, \tag{2.19}$$

the so-called adjoint Legendre function of n-th degree and rank k of first type[1].

Differentiating equation (2.8) k times and substituting $z = y^{(k)}$, we arrive at the following equation /90

$$(1-x^2) z'' - 2x(k+1) z' + [n(n+1) - k(k+1)] z = 0,$$

which is satisfied by the derivative of the function $z = P_n^{(k)}(x)$. After making the substitution

$$z = (1-x^2)^{-\frac{k}{2}} P_n^{(k)}(x)$$

and the appropriate transformations, we arrive at an equation for adjoint Legendre functions which coincides with equation (2.8).

The property of the adjoint functions of Legendre which is basic for the theory of the potential is given by the theorem of orthogonality, which is expressed as follows:

$$\int_{-1}^{1} P_n^k(x) P_m^k(x) dx = \begin{cases} 0 & \text{if } n \neq m, \\ \dfrac{2}{2n+1} \dfrac{(n+k)!}{(n-k)!} & \text{if } n = m. \end{cases} \tag{2.20}$$

[1] In addition to Legendre polynomials, mathematicians also deal with the Legendre functions of second type, and with adjoint Legendre functions of second type.

The formulas for Legendre polynomials (2.11) and adjoint Legendre functions (2.19) enable us to arrive at an explicit expression for a spherical function of the n-th degree (see (2.6)):

$$Y_n(\varphi, \lambda) = \sum_{k=0}^{n} (A_{nk} \cos k\lambda + B_{nk} \sin k\lambda) \cos^k \varphi \frac{d^k P_n(\sin \varphi)}{(d \sin \varphi)^k}. \tag{2.21}$$

From equation (2.21) it is evident that the spherical functions represent linear combinations of the functions

$$\frac{d^k P_n(\sin \varphi)}{(d \sin \varphi)^k} \cos^k \varphi \cos k\lambda, \quad \frac{d^k P_n(\sin \varphi)}{(d \sin \varphi)^k} \cos^k \varphi \sin k\lambda, \tag{2.22}$$

which are called spherical harmonics.

We shall take two spherical harmonics which differ by at least one index, and integrate their product with respect to the surface of the sphere. Following simple calculations we obtain the following:

$$\int_0^{\pi} \int_0^{2\pi} P_n^k(\sin \varphi) P_m^s(\sin \varphi) \begin{matrix}\cos \\ \sin\end{matrix} k\lambda \cdot \begin{matrix}\cos \\ \sin\end{matrix} s\lambda \cdot \cos \varphi \, d\lambda \, d\varphi = 0. \tag{2.23}$$

Consequently, the whole family of spherical harmonics represents an orthogonal system of functions. /91

Let the function $f(\phi, \lambda)$ be finite, homogeneous and continuous on the surface of a sphere of unit radius. We shall assume that it can be represented in the form of a series[1].

$$f(\varphi, \lambda) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} P_n^k(\sin \varphi)(A_{nk} \cos k\lambda + B_{nk} \sin k\lambda), \tag{2.24}$$

the terms of which are general spherical functions which were determined earlier by formula (2.6).

If the series (2.24) converges uniformly, then in order to determine its coefficients A_{nk} and B_{nk} we can make use of the property of orthogonality of spherical harmonics. Multiplying the series (2.24) by the spherical harmonic

$$P_n^k(\sin \varphi) \cos k\lambda$$

[1] This expression was first pointed out by Laplace, but the proof of its convergence was given by Legendre and Dirichlet.

and integrating with respect to the surface of the sphere, we arrive at the following equation by virtue of equation (2.23), following the appropriate operations:

$$A_{nk} = \frac{(2n+1)(n-k)!}{2\pi\delta_k (n+k)!} \int_0^{\pi} \int_0^{2\pi} f(\varphi, \lambda) P_n^k (\sin\varphi) \cos k\lambda \cos\varphi \, d\varphi \, d\lambda, \tag{2.25}$$

where $\delta_0 = 2$, $\delta_1 = \delta_2 = \ldots = 1$. In precisely the same way we find that

$$B_{nk} = \frac{(2n+1)(n-k)!}{2\pi\delta_k (n+k)!} \int_0^{\pi} \int_0^{2\pi} f(\varphi, \lambda) P_n^k (\sin\varphi) \sin k\lambda \cos\varphi \, d\varphi \, d\lambda. \tag{2.26}$$

It should be noted that by reason of the uniform convergence of the series (2.24), the expansion of the function is unique and does not depend upon the method by which it is derived.

In order to obtain a clearer idea of the expansion of the given function in series of spherical functions, we may compare the latter with the Fourier trigonometric series. With variation from 0 to 2π in the argument in the Fourier series, the n-th term of the series becomes zero 2n times. The routes of the n-th term are reflected in equidistant points on the circle, and on passage through the routes the terms of the series change sign. We find an analogous picture in the case of expansions in spherical functions. The spherical harmomics can be broken down into three categories. /92

For k = 0 there is only one harmonic $P_n^0 (\sin\phi) = P_n (\sin\phi)$, which becomes zero at n symmetric parallels to the equator. These parallels break up the surface of the sphere into zones, where the Legendre polynomials alternately take on positive and negative values. Thus, the harmonics corresponding to K = 0 are called zonal harmonics (Figure 15).

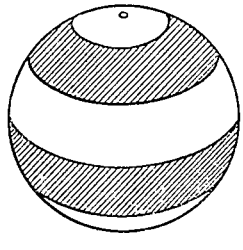

Figure 15.

If k = n, the attached Legendre functions $d^n P_n(x)/dx^n$ become constants and the spherical harmonics are reduced to:

$$\cos^n \varphi \cos n\lambda, \qquad \cos^n \varphi \sin n\lambda.$$

From this it is evident that the spherical harmonics become zero at the meridians for which

$$\cos n\lambda = 0 \quad \text{or} \quad \sin n\lambda = 0.$$

The surface of the sphere is divided into 2n spherical sectors by these meridians; in these sectors the spherical harmonics are alternately positive and negative. For this reason, the spherical harmonics in question are referred to as "sectorial" (Figure 16).

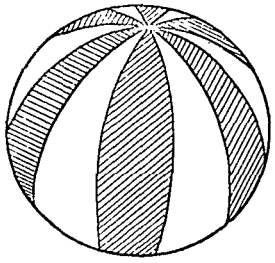

Figure 16.

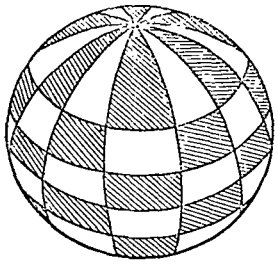

Figure 17.

Finally, in the intermediate case when $0 < k < n$ the spherical harmonics become zero at 2k meridians, with

$$\cos k\lambda = 0 \text{ if } \sin k\lambda = 0$$

and n -k parallels, defined by the routes of the equation /93

$$\frac{d^k P_n(\sin\varphi)}{(d\sin\varphi)^k} = 0.$$

The distribution of positive and negative values of spherical harmonics is illustrated in Figure 17. Such harmonics are referred to by the term "tesseral"[1].

§ 3. Expansion of the Potential of a Solid Body in Series in Spherical Functions.

Let us consider the gravitational potential of a solid body beginning with the assumptions formulated in § 1, Chapter 2. Utilizing the symbols employed earlier, for the potential we have the following expression:

$$U(x, y, z) = f\iiint \frac{\varkappa\, d\tau}{\Delta}, \tag{3.1}$$

where

$$\Delta = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}. \tag{3.2}$$

If we transfer to spherical coordinates,

$$x = r\cos\varphi\cos\lambda, \quad y = r\cos\varphi\sin\lambda, \quad z = r\sin\varphi,$$
$$\xi = r'\cos\varphi'\cos\lambda', \quad \eta = r'\cos\varphi'\sin\lambda', \quad \zeta = r'\sin\varphi'$$

and substitute /94

$$\cos\gamma = \frac{x\xi + y\eta + z\zeta}{rr'} = \sin\varphi\sin\varphi' + \cos\varphi\cos\varphi'\cos(\lambda - \lambda'), \tag{3.3}$$

[1] The classification and terminology relating to spherical functions were first devised by English mathematicians.

then for the distance between a moving point on the body and an external point P (x, y, z), we arrive at the expression

$$\Delta = \sqrt{r^2 + r'^2 - 2rr' \cos\gamma}.$$

Now let us expand the quantity $1/\Delta$, which appears as part of the integral expression of formula (3.1), into series in Legendre polynomials, limiting ourselves to the case $r' < r$. Recalling the form of the generating function of (2.16), we immediately arrive at

$$\frac{1}{\Delta} = \frac{1}{r}\left\{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\gamma\right\}^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\gamma). \tag{3.4}$$

Now let us study the convergence of series (3.4). First of all we shall estimate the absolute magnitude of the Legendre polynomial. If, for the integration interval in formula (2.14) we take the circle whose radius is $\sqrt{x^2 - 1}$ and whose center is at point $z = x$, whose equation has the form

$$z = x + \sqrt{x^2 - 1}\exp i\varphi, \quad -\pi < \varphi < \pi,$$

then, following the appropriate operations, the integral of (2.14) can be transformed to the following Laplace formula:

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} (x + \sqrt{x^2 - 1}\cos\varphi)^n d\varphi. \tag{3.5}$$

Formula (3.5) yields the following

$$|P_n(x)| \leqslant \frac{1}{\pi}\int_0^{\pi} |x + i\sqrt{1 - x^2}\cos\varphi|^n d\varphi$$

or, since in the given case $|x| < 1$, the preceding inequality can be written in the following form:

$$|P_n(x)| \leqslant \frac{1}{\pi}\int_0^{\pi} [1 - (1 - x^2)\sin^2\varphi]^{\frac{n}{2}} d\varphi, \tag{3.6}$$

from which it is evident that with $n > 0$ and for any value of $|x| < 1$, the following inequality is justified: /95

$$|P_n(x)| < 1. \tag{3.7}$$

Now let us go to an estimate of the absolute value of the common term of

series (3.4). On the basis of the inequality (3.7) we have

$$\left|\frac{r'^n}{r^{n+1}}P_n(\cos\gamma)\right|<\frac{r'^n}{r^{n+1}}.$$

But, by assumption, $\frac{\mathbf{r}'}{\mathbf{r}} = q < 1$, and therefore we finally arrive at

$$\left|\frac{r'^n}{r^{n+1}}P_n(\cos\gamma)\right|<\frac{q^n}{r}. \tag{3.8}$$

Since the series

$$\sum_{n=0}^{\infty}\frac{q^n}{r}$$

converges absolutely, series (3.4) possesses this same property. Moreover, on the basis of a well known theorem in the analysis of functional series, series (3.4) is found to be uniformly convergent. Thus, the assumptions which were made in the preceding paragraph apply equally in the present instance.

If we replace the quantity $1/\Delta$ in formula (3.1) with its value from (3.4), we arrive at the following expression:

$$U = f\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\iiint r'^n P_n(\cos\gamma)\varkappa\, d\tau. \tag{3.9}$$

In order to obtain a definitive expression for the series which represents the potential U, it is necessary to transform the Legendre polynomial $P_n(\cos\gamma)$, making use of a summation theorem which has the following form:

$$P_n(\cos\gamma) = \sum_{k=0}^{n}\frac{2(n-k)!}{\delta_k(n+k)!}P_n^k(\sin\varphi)P_n^k(\sin\varphi')\cos k(\lambda-\lambda'), \tag{3.10}$$

where, as above, $\delta_0 = 2$, $\delta_1 = \delta_2 = \ldots = 1$.

The basic idea involved in the derivation of formula (3.10) is as follows. /96 Owing to the linearity of the Laplace equation, a spherical function subjected to any linear orthogonal transformation of coordinates will satisfy the Laplace equation: in other words, the transformation merely produces a new spherical function. In particular, for any arbitrary rotation of the system of coordinates, the initial spherical function is transformed into a new function in the new coordinates.

The Legendre polynomial $P_n(\cos\gamma)$ can be regarded as a transformed

spherical function in a system of coordinates whose z-axis passes through the point (r', ϕ', λ'). Then, according to what has been said, the polynomial $P_n(\cos \gamma)$, corresponding to the new system of coordinates, must be a spherical function in the old system of coordinates (ϕ, λ) as well: in other words, there must exist values of the coefficients A_{nk} and B_{nk} which are independent of ϕ and λ:

$$P_n(\cos\gamma) = \sum_{k=0}^{n} P_n^k(\sin\varphi)\,(A_{nk}\cos k\lambda + B_{nk}\sin k\lambda). \tag{3.11}$$

It is obvious that the coefficients A_{nk} and B_{nk} are functions of ϕ' and λ'. But by reason of the symmetry of (3.3) with respect to ϕ, λ and ϕ', λ', formula (3.11) must have the following form:

$$P_n(\cos\gamma) = \sum_{k=0}^{n} h_k P_n^k(\sin\varphi)\,P_n^k(\sin\varphi')\cos k(\lambda - \lambda'), \tag{3.12}$$

where h_k represents certain numerical coefficients.

These coefficients can be calculated from the following formula:

$$h_k = \frac{2(n-k)!}{\delta_k (n+k)!}. \tag{3.13}$$

In this way we arrive at the formula (3.10), which represents the summation theorem of spherical functions. This formula will be used below in order to transform the expansion of the potential (3.9).

A detailed proof of this theorem may be found in [45] and [46].

Substituting for $P_n(\cos \gamma)$ in equation (3.9) its expression as obtained from formula (3.10), we arrive at the following: /97

$$U = f\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{k=0}^{n}\frac{2(n-k)!}{\delta_k(n+k)!}P_n^k(\sin\varphi)\times$$
$$\times\left[\iiint r'^n P_n^k(\sin\varphi')\cos k\lambda'\cdot\varkappa\,d\tau\cdot\cos k\lambda + \right.$$
$$\left. + \iiint r'^n P_n^k(\sin\varphi')\sin k\lambda'\cdot\varkappa\,d\tau\cdot\sin k\lambda\right].$$

Introducing the notations

$$\left.\begin{aligned} C_{nk} &= \frac{2(n-k)!}{\delta_k(n+k)!}\iiint r'^n P_n^k(\sin\varphi')\cos k\lambda'\cdot\varkappa\,d\tau,\\ D_{nk} &= \frac{2(n-k)!}{\delta_k(n+k)!}\iiint r'^n P_n^k(\sin\varphi')\sin k\lambda'\cdot\varkappa\,d\tau \end{aligned}\right\} \tag{3.14}$$

and observing, moreover, that these coefficients depend only upon the shape of the body and upon the distribution of mass within the body, we can present the preceding expression in this form:

$$U = f \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{k=0}^{n} P_n^k(\sin \varphi)(C_{nk} \cos k\lambda + D_{nk} \sin k\lambda). \tag{3.15}$$

Another approach, based on equations (2.19) and (2.21), is to replace in equation (3.15) the linear combination of spherical harmonics with the general spherical function. This results in

$$U = f \sum_{n=0}^{\infty} \frac{Y_n(\varphi, \lambda)}{r^{n+1}}. \tag{3.16}$$

We shall calculate the first terms occurring in the expansion of equation (3.16), making use of the series (3.9).

If n = 0, then, in correspondence with (2.12) P_0 (cos γ) = 1, and the integral

$$\iiint \varkappa \, d\tau = M$$

gives the mass M of the gravitating body. Since the first term will be equal to fM/r, we are in possession of the Newtonian potential of point mass.

If n = 1, then, by reason of equation (2.12),

$$P_1(\cos \gamma) = \frac{1}{rr'}(x\xi + y\eta + z\zeta),$$

the second term of the expansion can be written as follows: /98

$$\frac{f}{r^3}\left[x \iiint \xi \varkappa \, d\tau + y \iiint \eta \varkappa \, d\tau + z \iiint \zeta \varkappa \, d\tau\right].$$

In this expression the triple integrals represent the static moments, which, as is well known from mechanics, are equal to

$$\iiint \xi \varkappa \, d\tau = Mx_c, \quad \iiint \eta \varkappa \, d\tau = My_c, \quad \iiint \zeta \varkappa \, d\tau = Mz_c, \tag{3.17}$$

where x_c, y_c, z_c are the coordinates of the center of masses of the gravitating body.

If the origin of the coordinate system placed at the center of inertia of

the gravitating body, then the second term in the expansion of the potential becomes zero. In the general case this second term is written as follows:

$$\frac{fY_1}{r^3} = \frac{fM}{r^3}(xx_c + yy_c + zz_c).$$

If n = 2, then on the basis of equation (2.12) and (3.9), we have the following:

$$r^2r'^2P_2(\cos\gamma) = r^2r'^2\left(\frac{3}{2}\cos^2\gamma - \frac{1}{2}\right) =$$
$$= \frac{x^2}{2}(2\xi^2 - \eta^2 - \zeta^2) + \frac{y^2}{2}(2\eta^2 - \xi^2 - \zeta^2) +$$
$$+ \frac{z^2}{2}(2\zeta^2 - \xi^2 - \eta^2) + 3yz\eta\zeta + 3xz\xi\zeta + 3xy\xi\eta,$$

and then the third term of the expansion (3.9) can be written as follows:

$$\frac{fY_2}{r^5} = \frac{f}{r^5}\left\{\frac{x^2}{2}\iiint(2\xi^2 - \eta^2 - \zeta^2)\varkappa\,d\tau + \right.$$
$$+ \frac{y^2}{2}\iiint(2\eta^2 - \xi^2 - \eta^2)\varkappa\,d\tau + \frac{z^2}{2}\iiint(2\zeta^2 - \xi^2 - \eta^2)\varkappa\,d\tau + \qquad (3.18)$$
$$\left. + 3yz\iiint\eta\zeta\varkappa\,d\tau + 3xz\iiint\xi\zeta\varkappa\,d\tau + 3xy\iiint\xi\eta\varkappa\,d\tau\right\}.$$

Introducing the moments of inertia with respect to coordinate axes /99

$$\left.\begin{aligned} A &= \iiint\varkappa(\eta^2 + \zeta^2)\,d\tau, \\ B &= \iiint\varkappa(\xi^2 + \zeta^2)\,d\tau, \\ C &= \iiint\varkappa(\xi^2 + \eta^2)\,d\tau \end{aligned}\right\} \qquad (3.19)$$

and the centrifugal moments (products of inertia)

$$D = \iiint\varkappa\eta\zeta\,d\tau, \quad E = \iiint\xi\zeta\varkappa\,d\tau, \quad F = \iiint\varkappa\xi\eta\,d\tau, \qquad (3.20)$$

in place of (3.18) will have this form

$$\frac{fY_2}{r^5} = \frac{f}{2r^5}\{(B + C - 2A)x^2 + (A + C - 2B)y^2 +$$
$$+ (A + B - 2C)z^2 + 6Dyz + 6Exz + 6Fxy\}.$$

If the main axes of inertia are taken as coordinate axes, then D = E = F = = 0, and, therefore,

$$\frac{fY_2}{r^5} = \frac{f}{2r^5}\{x^2(B+C-2A) + y^2(A+C-2B) + z^2(A+B-2C)\}. \tag{3.21}$$

In the remaining portion of this text we assumed that the coordinate axes are selected in this manner, unless otherwise stated.

Limiting ourselves to the terms of the expansion thus found, we can formulate the following approximate expression for the gravitational potential of the body:

$$U = \frac{fM}{r}\left[1 + \frac{1}{r^2}(xx_c + yy_c + zz_c)\right] + \\ + \frac{f}{r^5}[(B+C-2A)x^2 + (A+C-2B)y^2 + (A+B-2C)z^2]. \tag{3.22}$$

§ 4. Potential of Terrestrial Gravitation.

The gravitational potential of the earth can be represented in the form of series (3.15), or, following certain simple transformation in the following form:

$$U = \frac{fM}{r}\left\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n} I_{nm}\left(\frac{R_0}{r}\right)^n P_{nm}\left(\frac{z}{r}\right)\cos m(\lambda - \lambda_{nm})\right\}, \tag{4.1}$$

where M is the mass of the earth, R_0 is the mean equatorial radius of the earth, /100 and I_{nm} and λ_{nm} are constants which depend upon the geometry of the masses of the earth. In formula (4.1) the index n assumes all real values beginning with 2: in other words, it is assumed that the origin of coordinates coincide with the center of mass of the earth, while the equatorial plane of the earth. is taken as the basic plane.

When the central axes of inertia are taken the coordinate axes, formula (4.1) assumes its simplest form. However, it is normal in geography that the x-axis should be made to pass the intersection of the Greenwich meridian and the equator; and if this is followed, then all of the terms in formula (4.1) are preserved, in particular the harmonics $P_{21}\cos(\lambda-\lambda_{21})$. From equation (4.1) it is evident that determining the parameters of the earth gravitational field reduces to finding the coefficients I_{nm}, λ_{nm}.

There are several essentially different methods for finding the gravitational field of the earth, or the shape of the earth. The methods of mechanics and the gravimetric methods are the most prominent.

The gravimetric method for determining the potential or the shape of the earth is based on measuring the acceleration of gravity at various points on the

earth's surface. A number of scientists, in past years, have proposed so-called formulas for the distribution of gravity. The most recent attempts in this direction have been made by I.D. Zhongolovich [50], Heiskanen [51], Uotilla [52], etc.

The work of these men has been based upon a large volume of observational data. One may point out, in particular, that I. D. Zhongolovich made use of 26,000 gravimetric points. In evaluating the accuracy of gravimetric methods, it was pointed out that they are less reliable when it is question of determining the coefficients tesseral harmonics. For example, the value of the geographic longitude of measure semi-axis of the terrestrial ellipsoid, as arrived at by the various writers, within the limits of -25° and +38°. A detailed presentation of the gravimetric method may be found in the book by I. L. Grushinskiy [53].

Of the method of celestial mechanics, in use before the launching /101
of artificial earth satellites, we should point out the following two. The first of these is based on analysis of the periodic inequalities observed the astronomical longitude and latitude of the moon which result from the spheroidal shape of the earth (we refer to the inequalities dealt with by Delaunay's theory of motion [54]). The second method is based on an analysis of a perculiarity of the rotational motion of the earth. The coefficients in the precession formulas of the terrestrial axis enable us to find the dynamic compression -- that is, the quantity (C - A)/C. Next, if we assume that the earth has a spheroidal shape (A = B), with the help of formula (3.28) we are able to determine the coefficient for the second zonal harmonic of the terrestrial potential.

The various other astronomical methods for determining the terrestrial potential are only of historical history at the present time, since observations of artificial earth satellites afford far greater accuracy in this connection. Coefficients for zonal and tesseral harmonics are determined on the basis of the secular and long-period perturbations observed in the orbital elements of such satellites. The basic difficulties are encountered in connection with the coefficients of tesseral harmonics; here it is along period (24-hour, 12-hour, etc.), inequalities (which are of small amplitude) which must be considered rather than secular variations. As many as 90 unknown parameters must be included in the general formulation of the conditional equations (the method of least squares is usually applied here). In order to obtain the most accuracy values possible for the coefficients I_{nm}, λ_{nm}, one must also include in these equations the coordinates of the tracking stations, the coefficient of light pressure, more than ten parameters representing the earth's atmosphere, and so on. The problem of determining the earth's gravitational field has occupied a great many investigators [55-60]. The results obtained by /102
I. Kozai [61] may be cited by way of example. Kozai's figures for the coefficients of zonal harmonics are given in Table 1; his figures for the coefficients of tesseral harmonics are given in Table 2.

From the results given here, it is quite evident that the coefficients of the expansion in series in terms of spherical functions fall off extremely

slowly[1]. Consequently, diminution in the amplitude of perturbations in the coordinates of artificial earth satellites, or in the elements of their orbits, may occur basically as a result of the coefficients of form $(R_0/r)^n$: in other words, along with increase in the major semi-axes of the satellite orbits. We are thus faced with the very complicated mathematical problem of constructing more rapidly converging series, representing the potential of terrestrial gravitation. In the theory of the shape of the earth use has been made of the expansion of the potential in series of Lamé functions [62, 63]; however, no radical improvement of this method has so far been achieved. It seems to the present writers that the most promising approach would be to improve the convergence of series in spherical functions, as is done in the theory of trigonometric series [64]. Unfortunately no investigations have been made along this line.

TABLE 1

n	$J_n \cdot 10^6$	n	$J_n \cdot 10^6$
2	1082,48 ±0,04	6	0,39 ±0,09
3	−2,566±0,012	7	−0,469±0,021
4	−1,84 ±0,09	8	−0,02 ±0,07
5	−0,063±0,019	9	0,114±0,025

TABLE 2

n	m	$J_{nm} \cdot 10^6$	λ_{nm}
2	2	2,32 ±0,30	−37°,5± 5°,6
3	1	3,95 ±0,36	22° ±11°
3	2	0,41 ±0,21	31° ±14°
3	3	1,91 ±0,29	51°,3± 2°,9
4	1	2,64 ±0,44	163°,5± 6°,5
4	2	0,17 ±0,06	54° ±11°
4	3	0,046±0,035	−13° ±19°
4	4	0,056±0,030	50°,3± 6°,0

§ 5. Gravitational Potentials of the Other Bodies of the Solar System

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If we are going to launch artificial satellites orbiting around the other celestial bodies of the solar system, and if we are going to predict the motion of such satellites, it is necessary to know the gravitational fields of those bodies. This is particularly necessary in the case of the moon, Jupiter, Saturn, Uranus, etc., since these particular bodies deviate quite strongly from the spherical shape. Even a direct optical observation reveal that the

[1] Careful estimates of the convergence rate of the expansion of the potential have been made by K. V. Kholshevnikov [206], on the basis of various premises regarding the distribution of planetary density.

flattening of Jupiter amounts to 1/10, while that of Saturn amounts to 1/16[1]. The degrees of flattening observed in these planets are much greater than that of the earth, which amounts to only 1/298.2. Consequently, the effects on the motion of artificial satellites orbiting around these planets and coming under the influence of the central body will be quite pronounced. Astronomical studies made of the moon lead to the conclusion that this body possesses not only a perceptable degree of polar flattening, but also a pronounced equatorial flattening, which is the result of the tidal action of the earth. In this connection, it is relevant to point out that it is precisely on count of the elliptical character of the lunar equator, that our satellite always turns the same "face" toward the earth.

Below are given some characteristics of the fields of gravitation of a number of the major planets of the solar system, and also of the moon. As regards the major planets, astronomers have been able to obtain more or less reliable data on the gravitational potentials in those cases in which the planets possess sufficiently close satellites. Actually, for the planets we have been able to determine only the coefficients of zonal harmonics -- that is, we are able to write the gravitational potential in the following form:

$$U = \frac{fM}{r}\left[1 - \frac{2}{3}J\left(\frac{R}{r}\right)^2 P_2\left(\frac{z}{r}\right) + \frac{4}{15}K\left(\frac{R}{r}\right)^4 P_4\left(\frac{z}{r}\right)\right]. \qquad (5.1)$$

Mars. The parameters of the Martian gravitational field were determined through analysis of the secular motion of the right ascension of the orbits of Phobos and Deimos, which amount, respectively, to 158°, 5 ± 0°, 5 and -6°, 2795 ± 0°,0007 per year. On the basis of these data it was possible to make a /104 reliable determination only of the coefficients of the second zonal harmonic, since, on account of the insufficient accuracy of the observations, errors in determining the regression of the orbital nodes of the Martian satellites were of the same order as that associated with the coefficient for the fourth harmonic. The most complete informational data have been used by E. Woolard [65], who arrived at the following value for the coefficient of the second harmonic:

$$J = 0{,}002920 - 0{,}0058\frac{\delta R''}{R''}.$$

Here R'' is the angular radius of Mars expressed in seconds of an arc. The first term in this formula was calculated on the assumption that R'' = 4''.680 at a distance of one astronomical unit.

Jupiter. The parameters of a gravitational potential of Jupiter were determined by the same method used in the case of Mars. Here we used information on the motion of the satellites closest to the planet, the well known Galilean satellites. On the basis of observational data compiled by de Sitter

[1] By the term "flattening" we understand the ratio of the difference between the equatorial and the polar radii of the planet to its equatorial radius.

[66], we obtained the following value for the coefficient of the first zonal harmonic

$$J = 0.02206 \pm 0.00022.$$

The coefficient for the fourth harmonic was corrected on the basis of the motion of the ascending node of the orbit of the fifth satellite. For the coefficient K we obtained the following value

$$K = 0.00253 \pm 0.0041.$$

For the radius of Jupiter we accepted the value R" = 98".49. A certain unevenness has been observed in the values obtained for J. For example, R. Sampson [67] obtained the value J = 0.00227. This is explained by the fact that the masses of the satellites play a part in the conditional equations for determining J and K, and these particular quantities are not known with sufficient accuracy. For this reason it is necessary to introduce corrections to the satellite masses, the corrections being determined by the method of least squares along with the parameters J and K.

Saturn. The coefficients of the expansion of the potential of Saturn were determined on the basis of secular perturbances in the latitude of the node and of the pericenter of the inner satellites of this planet. On the basis of data collected by Jeffreys [68], we obtained the following values: /105

$$J = 0{,}02501 \pm 0{,}00003, \quad K = 0{,}00386 \pm 0{,}00026.$$

For the radius of Saturn we accepted the value R" = 8".625 at a distance of 9.53885 astronomical units. These particular figures, however, may be in error since we are not in possession of reliable information on the masses of the rings of Saturn.

The Moon. Analytical methods for determining the parameters of the moon gravitational field which are based on the study of its rotational motion lead to the following values for the parameters of the expansion of the lunar potential [69]:

$$c_{20} = (-0{,}238 \pm 0{,}016) \cdot 10^{-3}, \quad c_{22} = (0{,}06 \pm 0{,}08) \cdot 10^{-4}.$$

By processing selenodetic data, Goudas [70] has arrived at the following values for the coefficients of the expansion of the lunar potential:

$$c_{20} = -0{,}205 \cdot 10^{-3},$$
$$c_{30} = -0{,}863 \cdot 10^{-4},$$
$$c_{40} = +0{,}263 \cdot 10^{-3}.$$

We should naturally expect that the most accurate results would be obtained on the basis of methods based on the study of the perturbed motion of artificial satellites in orbit around the moon. This particular method has, in fact, been employed by E. L. Akim [71], who analyzed the trajectoral measurements made for the lunar satellite "Luna-10" during the period from 3 April 1966 through 30 May 1966. This period represents 460 revolutions of the moon. In his study Akim made use of the analytical theory of the motion of a lunar satellite, based on the effect of the earth and the sun upon the moon. This latter factor is a quite substantial one, since the secular perturbations caused by the earth and by the sun must strongly affect the longitude of ascending node and the argument of the pericenter of the "Luna-10", their values being as follows:

$$\Delta\Omega = -1^\circ, \quad \Delta\omega = -2^\circ,$$

while the regression of the node and the lines of aspides of this satellite caused by the shape of the moon are as follows:

$$\Delta\Omega = -7^\circ.7, \quad \Delta\omega = -11^\circ.8.$$

E. L. Akim represents the potential of lunar gravitation as follows: /106

$$U = \frac{\mu}{r}\left\{1 + \sum_{n=2}^{\infty}\sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^n [c_{nm}\cos m\lambda + d_{nm}\sin m\lambda] P_n^m(\sin\varphi)\right\},$$

where R is the mean radius of the moon, ϕ is the selenocentric latitude reckoned from the mean equator of the moon, λ is the selenocentric longitude reckoned from the zero meridian for a certain epoch t_0, and supplies the following figures for the coefficients c_{nm} and d_{nm}, on the basis of an analysis of observational data:

$$c_{20} = (-0.206 \pm 0.022)\cdot 10^{-3}, \quad d_{21} = (0.361 \pm 0.358)\cdot 10^{-5},$$
$$c_{21} = (0.157 \pm 0.358)\cdot 10^{-5}, \quad d_{22} = (-0.139 \pm 0.145)\cdot 10^{-5},$$
$$c_{22} = (0.140 \pm 0.012)\cdot 10^{-4}, \quad d_{31} = (-0.178 \pm 0.032)\cdot 10^{-4},$$
$$c_{30} = (-0.363 \pm 0.099)\cdot 10^{-4} \quad d_{32} = (-0.702 \pm 4.595)\cdot 10^{-6}.$$
$$c_{31} = (-0.568 \pm 0.026)\cdot 10^{-4},$$
$$c_{32} = (0.118 \pm 0.047)\cdot 10^{-4},$$
$$c_{40} = (0.333 \pm 0.270)\cdot 10^{-4},$$

E. L. Akim reminds the reader that these results are of a preliminary character.

CHAPTER III

METHODS FOR THE CONSTRUCTION OF INTERMEDIATE ORBITS

§ 1. Formulation of the Problem. /107

Many writers, in dealing with the theory of the perturbed motion of artificial earth satellites, have taken the Keplerian elliptical orbit as their standard of an unperturbed (intermediate) orbit, and have employed the classical methods of celestial mechanics in choosing a particular intermediate orbit. Quite successful attempts based on this approach have been made by D. Brower [73, 74], I. Kosai [75, 76], A. A. Orlov [77, 78], D. Ye. Okhtsinskiy [72], and certain others. The first two authors used the Delone-Zeipel method, one of the strongest and most promising.

The choice of a Keplerian intermediate orbit is advantageous in that the calculation of the perturbations can be made on the basis of long-established methods, which allow the use of expansions in series of the coordinates of unperturbed motion without any modifications. It must be remembered, however, that these classical methods were devised to deal with the motions of the natural bodies of the solar system, and with the perturbing forces affecting them. These methods are particularly effective in cases of small eccentricity and low orbital inclination, and when the interval of time under consideration corresponds to only a few hundred revolutions. But is it precisely such conditions which are very frequently absent when we have to deal with artificial earth satellites. For example, the time intervals involved correspond to many hundreds and thousands of revolutions. Many of the artificial earth satellites which have been launched have very eccentric orbits; the eccentricity on occasion may exceed the celebrated Laplace limit ($e = 0.667$), which marks the region of convergence of the series expansions of the coordinates (and hence also the region of convergence of the expansion of the perturbation function). /108 All this complicates the use of the classical methods, since it is necessary to retain a large number of terms in the series. Finally, in the theory of motion of artificial earth satellites we cannot limit ourselves to first-order perturbations, as is usually done in analytical theories of the motion of natural satellites, asteroids, and the like; at least several secular perturbations of the second, and sometimes of the third-order must be calculated. This makes the construction of a precise analytical theory extremely cumbersome. Even more serious difficulties are encounted in the case of artificial moon satellites, since here it is absolutely necessary to allow for the moon's triaxiality (this means that a number of longitudinal terms must be retained in the approximate expression of the moon's gravitational potential).

One alternative to the use of classical methods of the theory of perturbations is to employ non-Keplerian intermediate orbits; this would allow for the most significant irregularities in the motion of spacecraft. There are precedents for this in certain classical problems, as, for example, Hill's

variation curve in the theory of lunar motion [79], and the problem of two immobile centers for the limited three-bodied problem (the reader may refer to C. Charlier [28], H. Samter [80], and also the present writer [81, 82]).

R. Newton [83] appears to have made the first in this direction when he pointed out the possibility of using the problem of two immobile centers in connection with the motion of a satellite of a spheroidal planet. The idea was subsequently developed by the present writer [84]. B. Garfinkel [33] has suggested a quite original method for plotting an intermediate orbit; this method has not, however, been completely substantiated on the theoretical level. It is J. Vinti [40] who has supplied us with the most successful and versatile choice of an unperturbed orbit for artificial earth satellites. M. D. Kislik [47] has dealt in detail with this method of designing an intermediate orbit, for the partial case of a spheroidal earth which is symmetrical with respect to the equatorial plane. The intermediate orbit devised by R. Barrar [37] is also deserving of consideration. This orbit yields less accurate results than Vinti's in the case of low-altitude earth satellites, but, thanks /109 to its simplicity, it is readily adaptable in the case of high-altitude earth satellites, as well as artificial satellites of other planets. Proceeding on the basis of the classical problem of two immobile centers, Ye. P. Aksenov, Ye. A. Grebenikov and the present writer have been able to demonstrate (through a generalization of the problem referred to) that the integrable cases of Vinti, Barrar and Kislik are really either partial or limiting cases of a single more general problem [13, 85]. These writers made a detailed qualitative study of the orbits of this problem [86-91].

The essential technique employed in the studies referred to was to devise an approximating expression for the gravitational potential for an axisymmetric planet such as will permit integration of the problem in closed form, in quadratures. Stäckel's theorem (see § 5, Chapter I) is used in the search for integrable cases.

§ 2. The Problem of Two Immobile Centers.

As noted earlier, it was R. Newton who recommended the problem of two immobile centers as a basis for studying the motion of artificial earth satellites. Subsequently, the present writer [92], [84] and V. G. Degtyarev [93, 94, 95] made some studies in this area. At this point we shall go over the basic results obtained.

The problem of two immobile centers consists in the study of the motion of a passively gravitating material point under the Newtonian gravitational attraction of two immobile point masses. Euler was the first to reduce this problem to quadratures (for the plane case) [96], and subsequently solutions for three-dimensions were achieved by Lagrange [97] and Jacobi [98]. But a general solution became possible only following the exhaustive qualitative analysis begun by Charlier [28] and completed by Badalyan [99, 100, 101], and Tallqvist [102], who dealt with the plane case, and by V. M. Alekseyev [103], who dealt with the three-dimensional case. These efforts yielded a complete classification of all possible forms of motion as well as a genealogy of the various classes of orbits.

Let us consider the motion of a material point P under the influence of gravitating immobile centers C_1 and C_2, whose masses are m_1 and m_2, respectively. We shall choose our system of coordinates in such a way that the points C_1 and C_2 lie on the z-axis, while the origin of coordinates is at the center of their masses. We shall designate the distance C_1C_2 by 2c, and the coordinates of the moving point by the symbols x, y, z (Figure 18). Then the force function of the problem will be written as follows: /110

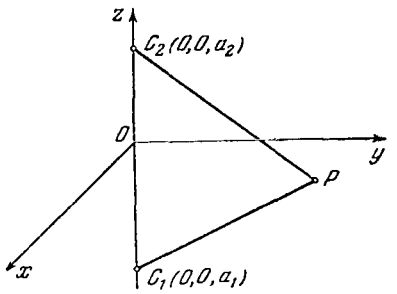

Figure 18.

$$U = \frac{fm_1}{r_1} + \frac{fm_2}{r_2}, \tag{2.1}$$

where the distances r_1 and r_2 from the moving point to the gravitating centers are equal to

$$\left.\begin{aligned} r_1 &= \sqrt{x^2 + y^2 + (z - a_1)^2}, \\ r_2 &= \sqrt{x^2 + y^2 + (z - a_2)^2}, \end{aligned}\right\} \tag{2.2}$$

while

$$a_1 = -\frac{2m_2c}{m_1 + m_2}, \qquad a_2 = \frac{2m_1c}{m_1 + m_2}. \tag{2.3}$$

Since we are interested in applying this problem to the study of the motion of artificial satellites, the parameters m_1, m_2 and c in expression (2.1) must be so chosen that the potential (2.1) will differ as little as possible from the potential of terrestrial gravitation.

With this purpose in mind we shall expand the reciprocals of the distances in series in Legendre polynomials, by using formula (2.16) Chapter 2. Then we obtain the following:

$$\left.\begin{aligned} \frac{1}{r_1} &= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{a_1}{r}\right)^n P_n\left(\frac{z}{r}\right), \\ \frac{1}{r_2} &= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{a_2}{r}\right)^n P_n\left(\frac{z}{r}\right), \end{aligned}\right\} \tag{2.4}$$

where r is defined by the formula /111

$$r = \sqrt{x^2 + y^2 + z^2}.$$

Substituting these series in formula (2.1), we obtain the following

expression for the function U:

$$U = \frac{f(m_1 + m_2)}{r}\left\{1 + \sum_{n=0}^{\infty} \frac{\gamma_n}{r^n} P_n\left(\frac{z}{r}\right)\right\}, \tag{2.5}$$

where

$$\gamma_n = \frac{m_1 a_1^n + m_2 a_2^n}{m_1 + m_2}. \tag{2.6}$$

According to the results obtained in § 4, Chapter 2, in a geocentric equatorial system of coordinates, the expression for the earth's gravitational potential (provided we neglect the longitudinal terms) has this form:

$$U_T = \frac{fM}{r}\left\{1 + \sum_{n=0}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n\left(\frac{z}{r}\right)\right\}, \tag{2.7}$$

where r is the geocentric radius-vector, M is the mass of the earth, and R is the mean equatorial radius of the earth.

A comparison of equations (2.5) and (2.6) reveals that, with a proper choice of the quantities m_1, m_2 and c, it is possible to achieve identity of the coefficients of the second and the third zonal harmonics of both series. The quantities m_1, m_2 and c must satisfy the following system of equations:

$$\left.\begin{aligned} m_1 + m_2 &= M, \\ 4m_1 m_2 c^2 &= M J_2 R^2, \\ 8m_1 m_2 (m_2 - m_1) c^3 &= M^3 J_3 R^3. \end{aligned}\right\} \tag{2.8}$$

As follows from equation (2.8), it is necessary that $m_1 m_2 < 0$, since the coefficient J_2 in the case of the earth is negative. Using equation (2.6) to determine the coefficient

$$\gamma_4 = \frac{16c^4 m_1 m_2}{M^4}(m_1^2 - m_1 m_2 + m_2^2) \tag{2.9}$$

and allowing for the fact that in the expansion of (2.7) the coefficient $J_4 > 0$, we see that the approximating potential (2.5) for the given choice of the constants m_1, m_2 and c, differs from the potential of terrestrial gravitation by the sign of the coefficient in the third harmonic. /112

But the fourth harmonic, in the case of certain osculating Keplerian

elements, yields secular inequalities, and for this reason approximation of the potential of the earth's force function, in connection with the problem of two immobile centers, can be used only for artificial satellites which are sufficiently remote from the planet. Nevertheless, we cannot exclude the possibility of obtaining good results when this method is applied to other planets.

There is still another means of determining the parameters of the approximating potential (2.5), whereby the coefficients γ_2 and J_2R^2 of expansions (2.5) and (2.7) coincide, while γ_4 assumes its minimal positive value. This method assures a minimal area in connection with the non-coincidence of the quantities γ_4 and J_4R^4. There is, however, an additional error involved in determining the perturbations from the third zonal harmonic. Since the coefficients J_3 and J_4 are of the same order, while the first-order perturbations from the third harmonic are periodic, the difference between the true motion and the intermediate motion will not exhibit secular variation, and, therefore, the suggested method will yield a better result than the first means of approximation.

Finally, we should point out the possibility of using approximate calculations for moon flight trajectories with the help of the problem of two immobile centers. Moon flight orbits, studied in detail by V. A. Yegorov, are calculated by numerical integration of the equations of the secular three-body problem -- (see equations (1.24) and (1.25), Chapter I). If n = 0, these equations become the same as the equations of motion in the problem of two immobile centers. If we use the methods proposed in [15, 28, 81], the circular three-body problem can be solved if the orbits of the problem of two immobile centers are taken as intermediate problems.

Let us proceed now to the integration of the equations of motion of this problem -- an operation most conveniently carried out with the use of spheroidal variables[1]. We shall first of all shift to a new system of rectangular coordinates, taking as the x-axis the line of centers, and placing the origin or coordinates midway between the gravitating centers. Then, applying the transformation of (3.29), Chapter I to the prolate spheroidal coordinates, we obtain, for the distance from the moving point to the centers /113.

$$r_1 = c\,(\operatorname{ch} v + \cos u), \tag{2.10}$$

$$r_2 = c\,(\operatorname{ch} v - \cos u), \tag{2.11}$$

and the force function will have the form

$$U = \frac{1}{cJ}\left[f\,(m_1 + m_2)\operatorname{ch} v - f\,(m_1 - m_2)\cos u\right], \tag{2.12}$$

[1] In the partial case of plane motion, regularization of the differential equations was first achieved by Thiele [104]. Thiele's method was applied to the three-dimensional case in [14].

where

$$J = \frac{r_1 r_2}{c^2} = \operatorname{ch}^2 v - \cos^2 u. \tag{2.13}$$

Then, according to (3.38), Chapter I, the equations of motion in spheroidal coordinates will be written as follows:

$$\left.\begin{aligned} &\frac{d}{dt}(J\dot{u}) - \sin u \cos u \cdot (\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \operatorname{sh}^2 v) = \frac{1}{c^2}\frac{\partial U}{\partial u}, \\ &\frac{d}{dt}(J\dot{v}) - \operatorname{sh} v \operatorname{ch} v \cdot (\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \sin^2 u) = \frac{1}{c^2}\frac{\partial U}{\partial v}, \\ &\frac{d}{dt}(\operatorname{sh}^2 v \sin^2 u \cdot \dot{w}) = 0. \end{aligned}\right\} \tag{2.14}$$

The system of equations (2.14) admits of two integrals: the integral of kinetic energy,

$$\frac{c^2}{2}[J(\dot{u}^2 + \dot{v}^2) + \dot{w}^2 \operatorname{sh}^2 v \sin^2 u] = U + h \tag{2.15}$$

and the integral of areas

$$\dot{w} \operatorname{sh}^2 v \sin^2 u = A. \tag{2.16}$$

Eliminating the cyclic coordinate w from equations (2.14), we obtain equations of motion in Routhian form:

$$\left.\begin{aligned} &\frac{d}{dt}(J\dot{u}) - \sin u \cos u \cdot (\dot{u}^2 + \dot{v}^2) = \frac{\partial U^*}{\partial u}, \\ &\frac{d}{dt}(J\dot{v}) - \operatorname{sh} v \operatorname{ch} v \cdot (\dot{u}^2 + \dot{v}^2) = \frac{\partial U^*}{\partial v}, \end{aligned}\right\} \tag{2.17}$$

where the transformed force function U* is defined by the formula

$$U^* = \frac{U}{c^2} - \frac{A^2}{2 \operatorname{sh}^2 v \sin^2 u}. \tag{2.18}$$

The integral of kinetic energy is transformed as follows: /114

$$J(\dot{u}^2 + \dot{v}^2) = 2(U^* + h). \tag{2.19}$$

By introducing a new independent variable τ, which is related to t by the differential equation

$$J\, d\tau = dt, \tag{2.20}$$

the system (2.17) is regularized, and assumes the following form:

$$\left.\begin{aligned} u'' - \frac{1}{2J}\frac{\partial J}{\partial u}(u'^2 + v'^2) &= J\frac{\partial U^*}{\partial u}, \\ v'' - \frac{1}{2J}\frac{\partial J}{\partial v}(u'^2 + v'^2) &= J\frac{\partial U^*}{\partial v}, \end{aligned}\right\} \tag{2.21}$$

where the primes denote differentiation with respect to τ. The variable τ we shall refer to as the "regularizing variable" or "regularized time". It is evident from equations (2.1) and (2.12) that the force function, and consequently also the right-hand members of the equations of motion (2.14) experience discontinuity for values of the coordinates which correspond to one of the gravitating centers. However, the transformed equations of motion (2.20) are free from this inconvenience. System (2.21) admits of a first integral

$$u'^2 + v'^2 = 2J\,(U^* + h), \tag{2.22}$$

corresponding to the integral of (2.19).

From equations (2.21) and (2.22) we obtain

$$\left.\begin{aligned} u'' &= \frac{\partial J}{\partial u}(U^* + h) + J\frac{\partial U^*}{\partial u}, \\ v'' &= \frac{\partial J}{\partial v}(U^* + h) + J\frac{\partial U^*}{\partial v}, \end{aligned}\right\} \tag{2.23}$$

or, substituting

$$W \doteq J\,(U^* + h), \tag{2.24}$$

we finally arrive at the following[1]:

$$u'' = \frac{\partial W}{\partial u}, \qquad v'' = \frac{\partial W}{\partial v}. \tag{2.25}$$

Integration of equations (2.25) yields /115

$$\left.\begin{aligned} \frac{1}{2}u'^2 &= -h\cos^2 u - \frac{f(m_1 - m_2)}{c^3}\cos u - \frac{A^2}{2\sin^2 u} + C', \\ \frac{1}{2}v'^2 &= h\,\mathrm{ch}^2 v + \frac{f(m_1 + m_2)}{c^3}\,\mathrm{ch}\,v - \frac{A^2}{2\,\mathrm{sh}^2 u} + C, \end{aligned}\right\} \tag{2.26}$$

[1] It should be noted that system (2.25) is not equivalent to system (2.17). Not all solutions of system (2.25) will be solutions of system (2.17). Extraneous solutions are excluded with the help of condition (2.27).

where C and C' are constants of integration. From (2.22) and (2.26) it follows that

$$C + C' = 0. \tag{2.27}$$

Introducing the new variables

$$\lambda = \operatorname{ch} v, \quad \mu = \cos u \tag{2.28}$$

and making allowance for (2.27), from (2.26) we obtain the following:

$$\left.\begin{aligned} \int_{\lambda_0}^{\lambda} \frac{d\lambda}{\sqrt{h\lambda^4 + \frac{f(m_1+m_2)}{c^3}\lambda^3 + (C-h)\lambda^2 - \frac{f(m_1+m_2)}{c^3}\lambda - \frac{A^2}{2} - C}} &= \\ &= \sqrt{2}(\tau - \tau_0), \\ \int_{\mu_0}^{\mu} \frac{d\mu}{\sqrt{h\mu^4 + \frac{f(m_1-m_2)}{c^3}\mu^3 + (C+h)\mu^2 - \frac{f(m_1-m_2)}{c^3}\mu - \frac{A^2}{2} - C}} &= \\ &= \sqrt{2}(\tau - \tau_1), \end{aligned}\right\} \tag{2.29}$$

where the symbols τ_0 and τ_1 denote constants of integration. Following inversion of the elliptic integrals of (2.29), λ and μ, and consequently also u and v, will be expressed in elliptic functions of Jacobi (or Weierstrass), the arguments of which will be linear functions of regularized time τ. In addition, from the integral of areas (2.16), which we shall represent in the following form:

$$\frac{dw}{d\tau} = A\left(\frac{1}{\sin^2 u} + \frac{1}{\operatorname{sh}^2 v}\right), \tag{2.30}$$

following substitution in place of u and v their expressions in terms of τ, we obtain the following as a result of integration:

$$w = A\int_{\tau_0}^{\tau}\left(\frac{1}{\sin^2 u} + \frac{1}{\operatorname{sh}^2 v}\right)d\tau + w_0. \tag{2.31}$$

Finally, from (2.20), with the help of quadrature we obtain /116

$$t - t_0 = \int_{\tau_0}^{\tau} J\,d\tau. \tag{2.32}$$

We shall refer to equation (2.32) as the equation of time (this equation is analogous to Kepler's equation, for the eccentric anomaly plays the role of a

regularizing variable in the two-body problem).

For the case of motion in a plane which contains the immobile centers, from equations (2.26) we obtain the following:

$$\left.\begin{aligned}\frac{d\lambda}{d\tau} &= \sqrt{2(\lambda^2-1)\left[h\lambda^2+\frac{f(m_1+m_2)}{c^3}\lambda+C\right]},\\ \frac{d\mu}{dt} &= \sqrt{2(\mu^2-1)\left[h\mu^2+\frac{f(m_1-m_2)}{c^3}\mu+C\right]}.\end{aligned}\right\} \tag{2.33}$$

If we introduce new variables ξ, η defined by the equalities

$$\lambda=\frac{1+\xi^2}{1-\xi^2}, \qquad \mu=\frac{1-\eta^2}{1+\eta^2}, \tag{2.34}$$

and if by the symbols λ_1, λ_2 and μ_1, μ_2 we designate respectively the routes of the equations

$$h\lambda^2+\frac{f(m_1+m_2)}{c^3}\lambda+C=0, \tag{2.35}$$

$$h\mu^2+\frac{f(m_1-m_2)}{c^3}\mu+C=0, \tag{2.36}$$

then, instead of (2.33) we have the following:

$$\frac{d\xi}{dt}=\sqrt{\frac{h}{2}[\xi^2(\lambda_2+1)-(\lambda_2-1)][\xi^2(\lambda_1+1)-(\lambda_1-1)]}, \tag{2.37}$$

$$\frac{d\eta}{dt}=\sqrt{-\frac{h}{2}[\eta^2(\mu_2+1)+(\mu_2-1)][\eta^2(\mu_1+1)+(\mu_1-1)]}. \tag{2.38}$$

Integrating (2.37) and (2.38) we arrive at the following:

$$\int_{\xi_0}^{\xi}\frac{d\xi}{\sqrt{\frac{h}{2}[\xi^2(\lambda_2+1)-(\lambda_2-1)][\xi^2(\lambda_1+1)-(\lambda_1-1)]}}=\tau-\tau_0, \tag{2.39}$$

$$\int_{\eta_1}^{\eta}\frac{d\eta}{\sqrt{-\frac{h}{2}[\eta^2(\mu_2+1)+(\mu_2-1)][\eta^2(\mu_1+1)+(\mu_1-1)]}}=\tau-\tau_1. \tag{2.40}$$

Inversion of the elliptic integrals (2.39)-(2.40) is accomplished in various different ways, depending upon the relationships between the routes of equations (2.35)-(2.36). The authors of [99-101] suggests the most interesting types of motion (from the point of view of practical application). All of these formulations are characterized by a negative value for total mechanical energy h. Let us consider one of them: /117

$$\lambda_2 > 1 > \lambda_1 > -1, \quad \mu_2 > 1 > \mu_1 > -1.$$

The integral of (2.29) can be represented as follows:

$$\int_a^{\xi} \frac{d\xi}{\sqrt{(a^2-\xi^2)(b^2+\xi^2)}} = \pm \sqrt{\frac{-2h(a^2+b^2)}{(1-a^2)(1+b^2)}}(\tau-\tau_0), \tag{2.41}$$

where a and b are defined from the following:

$$\lambda_1 = \frac{1-b^2}{1+b^2}, \quad \lambda_2 = \frac{1+a^2}{1-a^2}. \tag{2.42}$$

Substituting

$$\xi = a \cos \phi \tag{2.43}$$

equation (2.41) is transformed to the following form:

$$\int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \pm\sigma(\tau-\tau_0), \tag{2.44}$$

where

$$\sigma^2 = -\frac{2h(a^2+b^2)}{(1-a^2)(1+b^2)}, \quad k^2 = \frac{a^2}{a^2+b^2}, \tag{2.45}$$

whence

$$\varphi = \pm \operatorname{am}[\sigma(\tau-\tau_0), k]. \tag{2.46}$$

From equations (2.34), (2.43) and (2.46) we finally arrive at this equation:

$$\lambda = \frac{1+a^2\operatorname{cn}^2\sigma(\tau-\tau_0)}{1-a^2\operatorname{cn}^2\sigma(\tau-\tau_0)}. \tag{2.47}$$

Formula (2.47) affords a general solution to the first of the equations of (2.33). This solution will be periodic with respect to τ and will have the period

$$T = \frac{2}{\sigma}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{2}{\sigma}\mathbb{K}(k). \tag{2.48}$$

We proceed now to the inversion of the integral (2.40). We substitute /118

$$\mu_1 = \frac{1-\beta^2}{1+\beta^2}, \quad \mu_2 = \frac{1+\alpha^2}{1-\alpha^2} \tag{2.49}$$

and introduce the new variable ϕ, which is related with η by the following:

$$\eta = \beta \cos \phi. \tag{2.50}$$

As a result of transformations we obtain

$$\int_0^{\varphi} \frac{d\varphi}{\sqrt{1-\varkappa'^2 \sin^2 \varphi}} = \pm \sigma_1 (\tau - \tau_1). \tag{2.51}$$

In the latter formula the following designations were introduced:

$$\varkappa'^2 = \frac{\beta^2}{\alpha^2+\beta^2}, \quad \sigma_1^2 = -\frac{2h(\alpha^2+\beta^2)}{(1+\beta^2)(1-\alpha^2)}. \tag{2.52}$$

From equation (2.51) we find that

$$\eta = \beta \operatorname{cn} [i\sigma_1 (\tau - \tau_1), \varkappa']. \tag{2.53}$$

Shifting to the function of the real variable, we obtain

$$\eta = \frac{\beta}{\operatorname{cn} [\sigma_1 (\tau - \tau_1), \varkappa]}, \tag{2.54}$$

where

$$\varkappa^2 = \frac{\alpha^2}{\alpha^2+\beta^2}. \tag{2.55}$$

Finally, we arrive at

$$\mu = \frac{-\beta^2 + \operatorname{cn}^2 \sigma_1 (\tau - \tau_1)}{\beta^2 + \operatorname{cn}^2 \sigma_1 (\tau - \tau_1)}. \tag{2.56}$$

The general solution to the second of the equations of (2.33), which is given by formula (2.56), is periodic, having the period

$$T_1 = \frac{2}{\sigma_1} \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-\varkappa^2 \sin^2 \varphi}} = \frac{2}{\sigma_1} K(\varkappa). \tag{2.57}$$

From the formulas obtained for the general solution of the equations of (2.33), it follows that λ and μ are always periodic functions of τ. If, moreover, T and T_1 are commensurable, then the orbits will be closed following a certain number of revolutions, and, consequently, they will be periodic with respect to τ. Obviously, the problem in the plane case admits of ∞^2 periodic solutions. /119

If, however, T and T_1 are not commensurable, then the orbits will be periplegmatic, and will everywhere densely fill the region of possible motion. This particular result follows directly from the theorem stated in § 10, Chapter I. Since eigendegeneracy does not occur in this problem, the motion will be conditional-periodic, and the trajectory will everywhere densely fill a certain region of space u, v.

In a similar fashion we can arrive at explicit expressions for the spheroidal coordinates of the point, depending upon τ, and for three-dimensional motions. Unfortunately, no one has actually done this so far. The solution to the problem of two immobile centers can be sought-for in still another direction. For example, one may avoid elliptic functions entirely, replacing them with trigonometric series. In certain cases the method of Poincaré's minor parameter [92] can be used to good effect. R. K. Choudkhari [105] has made use of Steffensen's method [106] in the construction of polar series; this method enables us to obtain recurrent formulas for determining the coefficients, and for estimating the region of convergence of the series. The Steffensen method can be used conveniently in connection with electronic computers.

§ 3. Garfinkel's Method.

Affording the use of a Keplerian intermediate orbit, which does not represent the secular motions of the ascending node and the perigee, which are characteristic of artificial earth satellites, B. Garfinkel [107] breaks down the Hamiltonian of the perturbed problem in such a way that the unperturbed Hamiltonian H_0 will reflect the basic effects resulting from the non-spherical shape of the earth; in addition, this method makes it possible to integrate the problem in closed form.

If we introduce a geocentric equatorial system of spherical coordinates r, ϕ, λ, then the general solution to the problem in quadratures can be obtained by use of the Jacobi method, provided the force function has the following form (see (6.17), Chapter I):

$$U(r, \varphi) = f(r) + \frac{\Phi(\varphi)}{r^2}.$$

Let the unperturbed Keplerian orbit of an artificial earth satellite at the initial moment in time be characterized by the major semi-axis a, the eccentricity e and the inclination i. Then, according to B. Garfinkel, for the unperturbed portion of the Hamiltonian it is advantageous to adopt the function /120

$$H_0 = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2\cos^2\varphi} + \frac{p_\lambda^2}{r^2\cos^2\varphi}\right) - \\ - \frac{fM}{r}\left[1 + \frac{J_2R^2}{2r^2}\left(1 - \frac{3}{2}\sin^2 i\right) - \frac{3J_2R^2}{2ra(1-e^2)}\left(\sin^2\varphi - \frac{1}{2}\sin^2 i\right)\right]. \qquad (3.1)$$

Since the elements a, e, and i are considered to be constant, then, by comparing equation (3.1) with the preceding equation, we see that the simplified Hamiltonian (3.1) serves to describe the problem which is integrable in finite form.

In dealing with the motion of artificial earth satellites, Garfinkel considers only perturbations from the second zonal harmonic. Therefore, for the perturbation function H_1, we will have

$$H_1 = \frac{3}{2}fMJ_2R^2\left(\sin^2\varphi - \frac{1}{2}\sin^2 i\right)\left(\frac{1}{r^3} - \frac{1}{ar^2(1-e^2)}\right). \qquad (3.2)$$

If it is necessary to calculate the effect of zonal harmonics of higher order, the appropriate terms should be added to the perturbation function (3.2). From (3.1) and (3.2) it is evident that the Hamiltonian function H of the perturbed problem

$$H = H_0 + H_1$$

contains no initial values of the elements of the capillarian orbit. In the case of satellite orbits with small eccentricity, the perturbation function H_1 will remain of small magnitude on the order of $J_2 e$, at least for small intervals of time.

The Hamiltonian of the unperturbed (in the sense intended by Garfinkel) problem corresponds to a potential of the type of (6.17), Chapter I. Therefore, its solution follows directly from the results of § 6, Chapter I. The complete integral of the Hamilton-Jacobi equation is given by formula (6.21), Chapter I, provided that in place of f(r) and $\Phi(\phi)$ we substitute the corresponding values which in formula (3.1) represent the potential energy of the satellite. The complete integral is formulated as follows: /121

$$V = -\alpha_1 t + \int_{r_0}^{r}\sqrt{L(r)}\,dr + \alpha_2\lambda + \int_0^{\varphi}\sqrt{N(\varphi)}\,d\varphi, \qquad (3.3)$$

where r_0 is the geocentric distance of the artificial earth satellite at perigee, as determined from the equation L(r) = 0 for $\dot{r} = 0$, while the expressions L(r) and N(ϕ) under the radical are as follows:

$$L(r) = \frac{1}{r^3}\left\{2\alpha_1 r^3 - 2fMr^2 - \left[\alpha_2^2 - \frac{3fMJ_2R^2\sin^2 i}{2a(1-e^2)}\right]r - fMJ_2R^2\left(1 - \frac{3}{2}\sin^2 i\right)\right\}, \qquad (3.4)$$

$$N(\varphi) = \alpha_2 - \frac{3fMJ_2R^2}{a(1-e^2)}\sin^2\varphi - \frac{\alpha_3^2}{\cos^2\varphi}. \tag{3.5}$$

On the basis of the Jacobi theorem, the incomplete integral of the equations of motion of an artificial earth satellite can be written as follows:

$$\frac{\partial V}{\partial \alpha_1} = t - \beta_1 = \int_{r_0}^{r} \frac{dr}{\sqrt{L(r)}}, \tag{3.6}$$

$$\frac{\partial V}{\partial \alpha_2} = -\beta_2 = -\frac{1}{2}\int_{r_0}^{r} \frac{dr}{r^2\sqrt{L(r)}} + \frac{1}{2}\int_{0}^{\varphi} \frac{d\varphi}{\sqrt{N(\varphi)}}, \tag{3.7}$$

$$\frac{\partial V}{\partial \alpha_3} = -\beta_3 = \lambda - \alpha_3 \int_{0}^{\varphi} \frac{d\varphi}{\cos^2\varphi\sqrt{N(\varphi)}}, \tag{3.8}$$

where α_i represents the canonical distances, namely: α_1 is the total mechanical energy of the unity mass of the satellite; $\sqrt{\alpha_2}$ is the module of the kinetic potential, referred to the mass of the satellite; and α_3 is the constant of areas, corresponding to momentum with respect to the earth's axis of rotation. The conjugate canonical constants β_1, β_2, β_3, in line with the results of §7, Chapter I, represent, respectively, the moment of passage of the satellite through the perigee, the angular distance of perigee from the node (for $J_2 = 0$), and the longitude of the ascending node. If $J_2 = 0$, then the solution obtained determines the elliptical capillarian motion.

The problem of determining the canonical constants (elements) for Garfinkel's orbit can be solved quite simply. The interested reader may refer to T. E. Stern's book [107].

§ 4. Barrar's Problem /122

In the theory of remote earth satellites, and also in rough approximation, one can proceed from the potential suggested by R. Barrar [37, 108]. Barrar obtained an approximate expression for the earth's gravitational potential by combining the expansions of several functions in series in Legendre polynomials. But the mechanical interpretation of this approximation remains unclear. In the present study we have proceeded from the form of the equations which was obtained in § 6, Chapter I.

In this connection, the following assumptions were adopted:

1. The planet is a rigid body possessing an axis of dynamic symmetry: that is, A = B; and

2. The planet rotates around an axis which coincides with the smallest axis of the central ellipsoidal of inertia.

Let us adopt a planetocentric rectangular system of coordinates (with origin at the center of mass of the planet), the basic plane of which coincides with the equatorial plane of the planet, while the z-axis is directed along the planets rotational axis. Then, the gravitational potential of the planet is defined by formula (3.15), Chapter II. If we calculate the first three terms of this expansion, and substitute A = B in (3.20), Chapter II, we obtain

$$U = \frac{fM}{r} - \frac{f(A-C)}{2r^5}(x^2 + y^2 - 2z^2). \tag{4.1}$$

This approximate expression serves adequately in many satellite problems, since it contains the basic perturbed term (the second spherical harmonic). However, the problem with the potential (4.1) is not integrable in quadratures. In deriving an approximating potential, we therefore transform the coordinates as follows:

$$x = \xi, \quad y = \eta, \quad z - z_c = \zeta. \tag{4.2}$$

Here z_c is a quantity which can be selected subsequently.

The mechanical sense of the transformation (4.2) becomes clear if we recall certain facts from the geometry of masses. The ellipsoid of inertia, for an arbitrary selection of the pole, will in general be triaxial. However, for a certain choice of the pole, the ellipsoid of inertia may be an ellipsoid of rotation or a sphere. The pole for which the ellipsoid of inertia becomes /123
a sphere is called the spherical point.

Let us establish the conditions for which spherical points exist. Let A, B and C be the main central moments of inertia of the body, and let A', B' and C' be the moments of inertia, while D', E' and F' are the products of the interia for the pole at the point (x_c, y_c, z_c) with respect to the axes parallel to the principal central axes of inertia. Then (see [109]) the following relationships hold:

$$\left.\begin{aligned} A' &= A + M(y_c^2 + z_c^2), & D' &= My_cz_c, \\ B' &= B + M(x_c^2 + z_c^2), & E' &= Mx_cz_c, \\ C' &= C + M(x_c^2 + y_c^2), & F' &= Mx_cy_c. \end{aligned}\right\} \tag{4.3}$$

If the new pole is a spherical point, then any axis which passes through the pole must be a principal axis. Consequently, the following equality must hold:

$$D' = E' = F' = 0$$

or, by reason of (4.3)

$$y_c z_c = 0, \quad x_c z_c = 0, \quad x_c y_c = 0.$$

These equations are satisfied, for example, in the case

$$x_c = y_c = 0,$$

i.e., when the pole is taken on the z-axis. The third coordinate of the new pole is found from the condition

$$A' = B' = C'.$$

or in correspondence with (4.3)

$$A + Mz_c^2 = B + Mz_c^2 = C.$$

From this we find

$$z_c = \pm \sqrt{\frac{C - A}{M}}, \tag{4.4}$$

which is possible only if $C \geq A$.

Thus the existence of spherical points of inertia requires that the central ellipsoid of inertia shall be an oblate ellipsoid of inertia.

According to the results obtained in § 3, Chapter II, the condition /124

$$C - A > 0$$

holds for the earth, so that two spherical points can be designated for our planet, lying on its rotational axis. From formulas (3.20), Chapter II, we can determine numerical values for the z-coordinates of these points. Thanks to the numerous determinations thus far made of the coefficients of the expansion of the earth's potential in series in spherical functions, the coefficient for the second zonal harmonic is known with quite good accuracy. For z_c we assume the value

$$z_c = 209.9 \text{ km}.$$

Then, assuming that in the transformation (4.2) z_c is equal to the quantity obtained, instead of (4.1) we will have

$$U = \frac{fM}{\rho} + \frac{fMz_c\zeta}{\rho^3}, \tag{4.5}$$

where $\rho^2 = \xi^2 + \eta^2 + \zeta^2$. The transformation of coordinates have enabled us to eliminate all terms beyond the second zonal harmonic. However, since the origin of the new coordinate system does not coincide with the center of mass of the planet, the coefficient for the first spherical harmonic (see formula (3.20), Chapter II) will not be equal to zero.

Now let us transfer from the rectangular coordinates ξ, η, ζ to spherical coordinates ρ, ψ, λ:

$$\left.\begin{aligned} \xi &= \rho\cos\psi\cos\lambda, \\ \eta &= \rho\cos\psi\sin\lambda, \\ \zeta &= \rho\sin\psi. \end{aligned}\right\} \tag{4.6}$$

It is obvious that λ represents the right ascension of the satellite in the inertial system of coordinates, while the angle ψ, which differs only slightly from ϕ, may be referred to as the quasi-latitude (Figure 19).

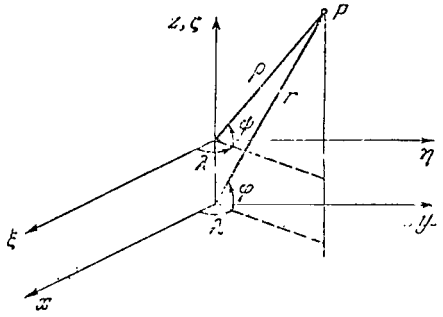

Figure 19.

The relationship between the two systems of spherical coordinates is defined by these equations:

$$\left.\begin{aligned} r &= \sqrt{\rho^2 - 2\rho z_c \sin\psi + z_c^2}, \\ \tan\phi &= \tan\psi - \frac{z_c}{\rho}\sec\psi \end{aligned}\right\} \tag{4.7}$$

The approximating potential (4.5) will have this form: /125

$$U = \frac{fM}{\rho} + \frac{fMz_c\sin\psi}{\rho^2}. \tag{4.8}$$

As follows from the Liouville theorem (see § 5, Chapter I), the equations of motion of the satellite in a force field (4.8) are integrable in quadratures.

In the present instance the equations of motion (3.19), Chapter I, are written in this form:

$$\left.\begin{aligned} \ddot{\rho} - \rho(\dot{\psi}^2 + \dot{\lambda}^2\cos^2\psi) &= \frac{\partial U}{\partial\rho}, \\ \frac{d}{dt}(\rho^2\dot{\lambda}\cos^2\psi) &= 0, \\ \frac{d}{dt}(\rho^2\dot{\psi}) + \frac{1}{2}\rho^2\dot{\lambda}^2\sin 2\psi &= \frac{\partial U}{\partial\psi}. \end{aligned}\right\} \tag{4.9}$$

From the second of these equations we can determine directly the integral of areas:

$$\rho^2\dot{\lambda}\cos^2\psi = c_1. \tag{4.10}$$

It is obvious that the system (4.9) admits of the integral of kinetic energy

$$\dot{\rho}^2 + \rho^2\dot{\psi}^2 + \rho^2\dot{\lambda}^2\cos^2\psi = 2U + c_2. \tag{4.11}$$

Multiplying the second equations of (4.9) by $2\rho^2\dot{\lambda}$, and the third by $2\rho^2\dot{\psi}$, and combining, we arrive at the following equation after integration:

$$\rho^4(\dot{\psi}^2 + \dot{\lambda}^2\cos^2\psi) = 2fMz_c\sin\psi + c_3. \tag{4.12}$$

This integral characterizes the variation in the module of the moment of momentum.

The existence of the three first integrals enables us to reduce the problem to quadratures. Leaving aside for the moment the qualitative study of the forms of motion, let us move on to the case which is most important for celestial ballistics, namely that in which $c_2 < 0$. The radius vector of the planet varies in this case according to Kepler's laws, but the angular coordinates as a function of time do not do so. Therefore the orbit of the satellite will be referred to quasi-elliptical.

§ 5. Solution of Barrar's Problem

Excluding from the integral (4.11) the quantities ψ and λ with the help of (4.12), we arrive at

$$\dot{\rho}^2 = 2U + c_2 - \frac{2fMz_c\sin\psi + c_3}{\rho^2}. \tag{5.1}$$

Allowing for equation (4.8), we transform equation (5.1) to the following form:

$$\dot{\rho}^2 = \frac{1}{\rho^2}(c_2\rho^2 + 2fM\rho - c_3). \tag{5.2}$$

The character of motion depends upon the roots of the equation

$$c_2\rho^2 + 2fM\rho - c_3 = 0. \tag{5.3}$$

If the arbitrary constants are such that the roots of equation (5.3) ρ_1 and ρ_2 ($\rho_2 > \rho_1$) are positive, then

$$\rho_1 = a\,(1 - e),\ \rho_2 = a\,(1 + e). \tag{5.4}$$

From the relationships

$$\rho_1\rho_2 = -\frac{c_3}{c_2},\quad \rho_1 + \rho_2 = -\frac{2fM}{c_2} \tag{5.5}$$

we find that

$$c_2 = -\frac{fM}{a},\quad c_3 = fMp, \tag{5.6}$$

where /127

$$p = a(1 - e^2). \tag{5.7}$$

In order to integrate equation (5.2), we make the following substitution:

$$\rho = a\,(1 - e\cos E). \tag{5.8}$$

Instead of equation (5.2) we will have

$$\dot{E} = \frac{n}{1 - e\cos E}, \tag{5.9}$$

where

$$n^2 = \frac{fM}{a^3}. \tag{5.10}$$

From (5.9) we obtain

$$E - e\sin E = n(t - T), \tag{5.11}$$

where T is the moment of passage of the satellite through the pericenter.

Assuming, as was done in the case of the two-body problem,

$$M = n(t - T), \tag{5.12}$$

we arrive at

$$E - e \sin E = M. \tag{5.13}$$

Instead of (5.12) we can make use of the relationship

$$M = M_0 + n(t - t_0), \tag{5.14}$$

where

$$M_0 = n(t_0 - T). \tag{5.15}$$

In place of time t we can introduce a new variable v, defined by the differential relationship

$$\rho^2 dv = \sqrt{fMp}\, dt, \tag{5.16}$$

then from (5.2) and (5.16) we obtain for the radius vector of the satellite the following expression:

$$\rho = \frac{p}{1 + e\cos v}. \tag{5.17}$$

From (5.8) and (5.17) we find

$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}. \tag{5.18}$$

The relationships obtained for the radius vector have the same form as those in the two-body problem; however, since we are not concerned here with plane orbits, but rather with space curves, the quantities introduced above cannot be interpreted in the same simple sense which they have in the case of the two-body problem. However, if we substitute $z_c = 0$ in the potential (4.8), then the quantities a, e and p, as well as certain others, assume the ordinary geometrical meaning. /128

Let us proceed now to a study of the quasi-latitude of the satellite, ψ. From the first integrals (4.10) and (4.11) we obtain

$$\rho^2 \dot{\psi}^2 = 2fMz_c \sin \psi + fMp - \frac{c_1^2}{\cos^2 \psi}. \tag{5.19}$$

With the help of (5.18) we transform equation (5.19) to the variable v:

$$\left(\frac{d\sin\psi}{dv}\right)^2 = -\frac{2z_c}{p}\sin^3\psi - \sin^2\psi + \frac{2z_c}{p}\sin\psi + 1 - \frac{c_1^2}{fMp}. \tag{5.20}$$

Substituting

$$c_1^2 = fMp\left(1 + \frac{2z_c}{p}\sin i\right)\cos^2 i \tag{5.21}$$

and

$$\delta = \frac{2z_c}{p}, \tag{5.22}$$

instead of (5.20) we have the following:

$$\left(\frac{d\sin\psi}{dv}\right)^2 = (\sin i - \sin\psi)[\delta\sin^2\psi + (1 + \delta\sin i)\sin\psi - \delta\cos^2 i + \sin i]. \tag{5.23}$$

Substituting the designation s = sin ψ, instead of equation (5.23) we arrive at the following:

$$\frac{ds}{dv} = \delta(s - s_1)(s - s_2)(s - s_3), \tag{5.24}$$

where s_1, s_2 and s_3 denote the roots of the polynomial

$$\left.\begin{aligned} s_1 &= \sin i, \\ s_{2,3} &= \frac{1}{2\delta}\left(-1 - \delta\sin i \pm \sqrt{1 - 2\delta\sin i + \delta^2(1 + 3\cos^2 i)}\right) \end{aligned}\right\} \tag{5.25}$$

We now transform equation (5.24) to a new variable u, defined as follows: /129

$$s = s_1 + (s_2 - s_1)\sin^2 u. \tag{5.26}$$

As a result of transformations, we arrive at the following equation:

$$\frac{du}{dv} = \sigma\sqrt{1 - k^2\sin^2 u}, \tag{5.27}$$

where

$$\left.\begin{aligned} \sigma &= \frac{1}{2}\sqrt{\delta(s_1 - s_3)}, \\ k &= \sqrt{\frac{s_1 - s_2}{s_1 - s_3}}. \end{aligned}\right\} \tag{5.28}$$

Integrating (5.27), we arrive at

$$u = \operatorname{am}(\tau, k), \tag{5.29}$$

where

$$\tau = \sigma(v + \omega), \tag{5.30}$$

where ω is an arbitrary constant. From equations (5.27) and (5.29), it follows that

$$s = s_1 + (s_2 - s_1)\operatorname{sn}^2\tau. \tag{5.31}$$

From equations (4.10), (5.21) and (5.22) we arrive at an equation for determining the longitude of the satellite:

$$\frac{d\lambda}{dt} = \frac{\sqrt{fMp(1+\delta\sin i)}\cos i}{\rho^2\cos^2\psi}, \tag{5.32}$$

from which it follows that

$$\frac{d\lambda}{ds} = \frac{\sqrt{fMp(1+\delta\sin i)}\cos i}{\rho^2\dot{s}\cos^2\psi},$$

or

$$\frac{d\lambda}{ds} = \frac{\sqrt{1+\delta\sin i}\cos i}{\cos^2\psi}\frac{dv}{ds}.$$

Finally, we arrive at

$$\frac{d\lambda}{ds} = \frac{\sqrt{1+\delta\sin i}\cos i}{(1-s^2)\sqrt{\delta(s_1-s)(s-s_2)(s-s_3)}}. \tag{5.33}$$

From (5.33), following integration we have /130

$$\lambda - \Omega = \frac{1}{2}\sqrt{\frac{1+\delta\sin i}{\delta}}\cos i\left\{\int_{s_0}^{s}\frac{ds}{(1-s)\sqrt{(s_1-s)(s-s_2)(s-s_3)}} + \right.$$
$$\left. + \int_{s_0}^{s}\frac{ds}{(1+s)\sqrt{(s_1-s)(s-s_2)(s-s_3)}}\right\}, \tag{5.34}$$

where Ω is the constant of integration.

With the help of equation (5.20), we transform the integrals which appear in (5.34) to normal Legendre form:

$$\lambda - \Omega = -\cos i \sqrt{\frac{1+\delta \sin i}{\delta (s_1 - s_3)}} \left\{ \frac{1}{1-s_1} \Pi(u, n', k) + \right. \\ \left. + \frac{1}{1+s_1} \Pi(u, n'', k) \right\}, \tag{5.35}$$

where the Π denotes elliptical integrals of the third type, while the parameters of these integrals are equal to

$$n' = \frac{s_1 - s_2}{1 - s_1}, \quad n'' = \frac{s_3 - s_1}{1 + s_1}. \tag{5.36}$$

§ 6. The Generalized Problem of Two Immobile Centers

In the preceding paragraphs we have considered various means of approximating the potential of a spheroidal planet. The studies made by Ye. P. Aksenov, Ye. A. Grebenikov and the present author [85] have shown that all approximating potentials which are based upon the Garfinkel potential represent either partial or limiting cases of the potential of the generalized problem of two immobile centers.

Let us examine the motion of a passively gravitating material particle under the gravitational attraction of two immobile centers, P_1 and P_2. Here we shall adopt a rectangular system of coordinates in which the z-axis lies on the line of centers P_1P_2.

The potential of the problem is defined by formula (2.1):

$$U = \frac{fM_1}{r_1} + \frac{fM_2}{r_2}, \tag{6.1}$$

where /131

$$\left. \begin{aligned} r_1 &= \sqrt{x^2 + y^2 + (z - a_1)^2}, \\ r_2 &= \sqrt{x^2 + y^2 + (z - a_2)^2}, \end{aligned} \right\} \tag{6.2}$$

M_1 and M_2 are the masses of the gravitating centers P_1, P_2, a_1 and a_2 are the distances from these centers to the origin of coordinates. If the distance between the centers is assumed to be a, while the origin of the coordinate system is taken at the center of inertia of the points P_1 and P_2,

then a_1 and a_2 are represented as follows:

$$a_1 = \frac{M_2 a}{M_1 + M_2}, \quad a_2 = -\frac{M_1 a}{M_1 + M_2}. \tag{6.3}$$

As regards their mechanical mean, the constants M_1, M_2, a_1 and a_2 should be regarded as real numbers. In this discussion, however, we shall not limit ourselves by this restriction, but shall consider a complex values for these quantities as well. The potential (6.1), however, must assume only real values. The function U we shall construct in such a way that its expansion in series in spherical functions will coincide with the expansion of the gravitational potential of a spheroidal planet.

In correspondence formula (2.4), we have

$$\left.\begin{aligned} \frac{1}{r_1} &= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{a_1}{r}\right)^n P_n\left(\frac{z}{r}\right), \\ \frac{1}{r_2} &= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{a_2}{r}\right)^n P_n\left(\frac{z}{r}\right), \end{aligned}\right\} \tag{6.4}$$

where $r^2 = x^2 + y^2 + z^2$. If we substitute these series in formula (6.1), we arrive at the following expansion of the potential:

$$U = \frac{fM}{r}\left[1 + \sum_{n=1}^{\infty}\frac{\gamma_n}{r^n} P_n\left(\frac{z}{r}\right)\right], \tag{6.5}$$

in which use has been made of the following designations:

$$M = M_1 + M_2, \quad \gamma_n = \frac{M_1 a_1^n + M_2 a_2^n}{M}. \tag{6.6}$$

As is apparent from equation (6.5), the force function U will be real, provided the constants M and γ_n are real for any whole number n. If the origin of coordinates is taken at the center of inertia of the planet, then by virtue of (3.20), Chapter II, in the expansion of the potential of the planet the first spherical harmonic disappears. For this reason, in the expansion (6.5) we require that $\gamma_1 = 0$. Thus, the potential (6.1) represents the gravitational potential of a certain solid body, provided that

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$$\gamma_1 = 0, \tag{6.7}$$

$$\operatorname{Im} \gamma_n = 0 \qquad (n = 2, 3, \ldots), \tag{6.8}$$

$$\operatorname{Im} M = 0. \tag{6.9}$$

Let us establish now what conditions must be satisfied by the quantities M_1, M_2, a_1, a_2, in order that the relationships (6.7) - (6.8) shall be in effect. With this in mind, we shift to the trigonometric form of the complex quantities M_1, M_2, a_1 and a_2, making the substitutions

$$\left.\begin{aligned} M_1 &= \rho_1 \exp i\psi_1, \quad M_2 = \rho_2 \exp i\psi_2, \\ a_1 &= R_1 \exp i\varphi_1, \quad a_2 = R_2 \exp i\varphi_2. \end{aligned}\right\} \tag{6.10}$$

Condition (6.7), on the basis of (6.10), is written in the form

$$\left.\begin{aligned} \rho_1 R_1 \cos(\psi_1 + \varphi_1) + \rho_2 R_2 \cos(\psi_2 + \varphi_2) &= 0, \\ \rho_1 R_1 \sin(\psi_1 + \varphi_1) + \rho_2 R_2 \sin(\psi_2 + \varphi_2) &= 0, \end{aligned}\right\} \tag{6.11}$$

while the relationships of (6.8) reduce to the following form:

$$\sin(\psi_1 + n\varphi_1) + \frac{\rho_2}{\rho_1}\left(\frac{R_2}{R_1}\right)^n \sin(\psi_2 + n\varphi_2) = 0. \tag{6.12}$$

From (6.11) we find the equation

$$\tan(\varphi_1 + \psi_1) = \operatorname{tg}(\varphi_2 + \psi_2), \tag{6.13}$$

in solving which we arrive at

$$\varphi_1 + \psi_1 = \varphi_2 + \psi_2 + \pi s, \tag{6.14}$$

where s is any whole number.

Assuming for the time being that $\cos(\phi_1 + \psi_1) \neq 0$, then from (6.11) and (6.14) we find that

$$\frac{\rho_2}{\rho_1} = \frac{R_1}{R_2}. \tag{6.15}$$

Then equation (6.12) is reduced to this form

$$\sin(\psi_1 + n\varphi_1) + \left(\frac{R_2}{R_1}\right)^{n-1} \sin(\psi_2 + n\varphi_2) = 0. \tag{6.16}$$

If $\phi_1 \neq 0$ or π, and if $\psi_1 \neq 0$ or π (i = 1, 2), then equation (6.16) for all integral values of n can be justified only on the condition /133

$$R_1 = R_2. \tag{6.17}$$

In that case, from equation (6.16) we have

$$\tan\ (\varphi_1 + \psi_1) = \operatorname{ctg}\left[(n-1)\frac{\varphi_1 + \varphi_2}{2}\right], \tag{6.18}$$

and the latter equation holds for all integral values only when

$$\varphi_1 + \varphi_2 = 0;\ 2\pi. \tag{6.19}$$

From this it follows that

$$\left.\begin{aligned} &\varphi_1 + \psi_1 = \frac{\pi}{2} + \pi s_1, \\ &\varphi_2 + \psi_2 = \frac{\pi}{2} + \pi s_2 \quad (s_1, s_2 = 0, 1, 2, \ldots). \end{aligned}\right\} \tag{6.20}$$

Combining equations (6.19) and (6.20), we have

$$\psi_1 + \psi_2 = \pi s_3 \quad (s_3 = 0, 1, 2, \ldots). \tag{6.21}$$

As follows from equations (6.15), (6.17), (6.19) and (6.20), M_1, M_2, a_1, a_2 must be mutually paired adjoint complex quantities.

In the second possible case

$$\left.\begin{aligned} &\varphi_i = 0 \quad \text{if} \quad \pi, \\ &\psi_i = 0 \quad \text{if} \quad \pi \quad (i = 1, 2), \end{aligned}\right\} \tag{6.22}$$

which affords integral values for the quantities under consideration -- in other words in this case we have arrived at the classical problem of two immobile centers.

In the preceding discussions we proceeded on the assumption that

$$\varphi_i + \psi_i \neq \pm\frac{\pi}{2} + 2\pi k_i \qquad (i = 1, 2).$$

But if these conditions are not met, then equation (6.12) does not hold for all integral values of n. Hence, the potential U assumes real values only in two cases. In the first case M_1 and M_2, and also a_1 and a_2 are mutually paired adjoint complex quantities: /134

$$\left.\begin{aligned} M_1 &= \frac{M}{2}(1+i\delta), \quad & M_2 &= \frac{M}{2}(1-i\delta), \\ a_1 &= c(\varkappa+i), & a_2 &= c(\varkappa-i), \end{aligned}\right\} \tag{6.23}$$

where M, c, δ, κ are real, and $i = \sqrt{-1}$.

Allowing for the condition (6.8), which gives $\kappa - \delta = 0$, from equation (6.23) we obtain

$$\left.\begin{aligned} M_1 &= \frac{M}{2}(1+i\delta), \quad & a_1 &= c(\delta+i), \\ M_2 &= \frac{M}{2}(1-i\delta), & a_2 &= c(\delta-i). \end{aligned}\right\} \tag{6.24}$$

In the second case, all of the constants M_1, M_2, a_1, a_2 are real numbers. Denoting the ratio of mass M_2 to the sum of masses M with the symbol γ from (6.3) we obtain

$$\left.\begin{aligned} M_1 &= M(1-\gamma), \quad & a_1 &= a\gamma, \\ M_2 &= M\gamma, & a_2 &= -a(1-\gamma). \end{aligned}\right\} \tag{6.25}$$

Thus the function U, defined by formula (6.1), is the potential of a rigid body only in the case in which the constants M_1, M_2, a_1, a_2 are assigned either by equations (6.24) or by equation (6.25).

If we apply the assumptions formulated at the beginning of § 5 to the earth, then the gravitational potential of the planet will be (§ 3, Chapter II)

$$U_r = \frac{fM}{r}\left[1 + \sum_{k=2}^{\infty} J_k \left(\frac{R}{r}\right)^k P_k\left(\frac{z}{r}\right)\right] \tag{6.26}$$

where M denotes the mass of the earth, J_k are certain constants, and R = = 6,378.1 km.

If in equation (6.1) we substitute the value of the parameters from equation (6.24), then the potential will be defined by the following equation:

$$U = \frac{fM}{2}\left\{\frac{1+i\delta}{r_1} + \frac{1-i\delta}{r_2}\right\}, \tag{6.27}$$

where

$$\left.\begin{aligned} r_1 &= \sqrt{x^2 + y^2 + [z - c(\delta + i)]^2}, \\ r_2 &= \sqrt{x^2 + y^2 + [z - c(\delta - i)]^2}, \end{aligned}\right\} \tag{6.28}$$

and its expansion in series in Legendre polynomials, /135

$$U = \frac{fM}{r}\left[1 + \sum_{k=0}^{\infty} \frac{\gamma_k}{r^k} P_k\left(\frac{z}{r}\right)\right] \tag{6.29}$$

will have the following coefficients:

$$\gamma_k = \frac{c^k}{2}[(1 + i\delta)(\delta + i)^k + (1 - i\delta)(\delta - i)^k]. \tag{6.30}$$

Comparing the expansions of (6.26) and (6.29), we see that the potential of two immobile centers represents approximately the gravitational field of the earth. Actually, for this purpose we have at our disposal the parameters which appear in equation (6.27), so that the coefficients in the first three terms of the expansions of (6.26) and (6.27) will coincide. To do this we set M equal to the mass of the earth, and determine c and δ from

$$c^2(1 + \delta^2) = -J_2R^2, \tag{6.31}$$

$$2\delta c^3(1 + \delta^2) = -J_3R^3, \tag{6.32}$$

solving these equations for c and δ, we arrive at

$$c = \frac{\sqrt{-J_3^2 - 4J_2^3}}{2J_2} R, \quad \delta = \frac{J_3}{\sqrt{-J_3^2 - 4J_2^3}}. \tag{6.33}$$

We shall also find the coefficient

$$\gamma_4 = c^4(1 + \delta^2)(1 - 3\delta^2). \tag{6.34}$$

A number of writers have obtained numerical values for the coefficients J_k in the expansion of the terrestrial gravitational potential [55-61]. Table 3 shows how little J_4 differs from the corresponding coefficient in the potential of the earth with the sampling of c and δ used.

Thus the approximating expression of the potential obtained from the generalized problem of two immobile centers offers a very good approximation of the earth's gravitational potential.

From Table 3 and formulas (6.33) it is evident that the linear quantity c introduced here is in fact the z-coordinate of the spherical point of inertia.

When $\delta = 0$, then formula (6.27) gives the partial case corresponding to M. D. Kislik's potential. This potential does not take into account the asymmetry of the earth with respect to the equatorial plane; however, the corresponding intermediate orbits are very close to those actually observed. /136

TABLE 3

$J_2 \cdot 10^6$	$J_3 \cdot 10^6$	$J_4 \cdot 10^6$	c (км)	$\delta \cdot 10^2$	$\frac{\gamma_4}{R^4} \cdot 10^6$	Source
—1082,2	2,3	2,1	209,8	—3,2	1,2	[56]
—1082,8	2,4	1,4	209,9	—3,4	1,2	[57]
—1082,5	2,4	1,7	209,8	—3,4	1,2	[60]
—1083,0	—	1,1	209,9	—	1,2	[59]
—1082,7	—	2,0	209,9	—	1,2	[58]

Now let us represent the potential of (6.1) in the form

$$U = fM\left[\frac{1-\gamma}{\sqrt{x^2+y^2+(z-\delta_1+\delta_1-a_1)^2}} + \frac{\gamma}{\sqrt{x^2+y^2+(z-\delta_1+\delta_1-a_2)^2}}\right], \tag{6.35}$$

where δ_1, which remains undefined, is a small quantity. Expanding the inverse distances in Taylor's series, and restricting ourselves to terms of the first order of smallness with respect to the quantities δ_1, a_1, a_2, we obtain

$$U \approx \frac{fM}{\rho}\left[1 - \frac{\delta_1(z-\delta_1)}{\rho^2}\right], \tag{6.36}$$

where

$$\rho^2 = x^2 + y^2 + (z-\delta_1)^2 \tag{6.37}$$

If we stipulate

$$\delta_1 = a\sqrt{\gamma(1-\gamma)} = R\sqrt{-J_2},$$

then formula (6.36) assumes the same form as the formula which defines Barrar's potential (see (4.5)). Consequently the Barrar potential is really a simplified expression for the force function of the problem of two immobile centers. In § 4 we clarified the meaning of the Barrar potential from the point of view of the geometry of masses. Proceeding from the problem of two immobile centers, one can regard the Barrar potential as being the gravitational potential of an immobile center and a "dipole"; i.e., essentially /137 the Barrar potential is a limiting variant of the potential of two gravitating centers.

Vinti has also proposed an approximate formula for the potential of terrestrial gravitation. If this formula is expressed in terms of a coordinate system which has been shifted along the earth's rotational axis, rather than in terms of a geocentric system, it then represents a partial case of the generalized problem of two immobile centers. Actually, in case of the complex distances $1/r_1$ and $1/r_2$ we can write

$$r_1 = \sqrt{\rho^2 - 2ic\zeta + i^2c^2}, \quad r_2 = \sqrt{\rho^2 + 2ic\zeta + i^2c^2},$$

where

$$\rho = \sqrt{x^2 + y^2 + (z - c\delta)^2}, \quad \zeta = z - c\delta.$$

We now expand $1/r_1$ and $1/r_2$ in power series of ic/ρ. Then, substituting these series in formula (6.27), we have

$$U = \frac{fM}{\rho}\left[1 + \sum_{n=1}^{\infty}(-1)^n c^{2n}\rho^{-2n}P_{2n}\left(\frac{\zeta}{\rho}\right) - c\delta\sum_{n=0}^{\infty}(-1)^n c^{2n}\rho^{-2n-1}P_{2n+1}\left(\frac{\zeta}{\rho}\right)\right] \tag{6.38}$$

The potential (6.38) coincides with Vinti's potential within the limits of accuracy imposed by the choice of quantities.

Thus, we see that all of the approximating expressions for the earth's potential proposed by various writers represent either partial or limiting cases of the force function of the generalized problem of two immobile centers. This is an important fact, inasmuch as the methodology devised for the classical problem of two immobile centers can, in its entirety, be applied to the study of the motion of artificial earth satellites. Doing so enables one to make use of results obtained earlier by the present writer [15] in the integration of the three dimensional generalized problem of two centers.

Now let us consider the integration of the differential equations of motion.

In oblate spheroidal coordinates (3.20), Chapter I, /138

$$\left.\begin{aligned} x &= c\,\mathrm{ch}\,v\sin u\cos\omega,\\ y &= c\,\mathrm{ch}\,v\sin u\sin\omega,\\ z &= c\delta + c\,\mathrm{sh}\,v\cos u \end{aligned}\right\} \tag{6.39}$$

the differential equations, in correspondence with (3.28), Chapter I, are written as follows

$$\left.\begin{aligned}
&\frac{d}{dt}(J\dot{u}) + (\dot{u}^2 + \dot{v}^2 - \dot{w}^2 \operatorname{ch}^2 v)\sin u \cos u = \frac{1}{c^2}\frac{\partial U}{\partial u},\\
&\frac{d}{dt}(J\dot{v}) - (\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \sin^2 u)\operatorname{sh} v \operatorname{ch} v = \frac{1}{c^2}\frac{\partial U}{\partial v},\\
&\frac{d}{dt}(\dot{w} \operatorname{ch}^2 v \sin^2 u) = 0,
\end{aligned}\right\} \tag{6.40}$$

while the function U, following transformation to real form, can be written as follows, with the help of Euler's well-known formulas:

$$U = \frac{fM}{c} \cdot \frac{\operatorname{sh} v - \delta \cos u}{\operatorname{sh}^2 v + \cos^2 u}. \tag{6.41}$$

Comparing (6.41) and (8.12), Chapter I, we observe that the potential of the generalized problem of two immobile centers admits of integration in quadratures.

The system (6.40) possesses two first integrals: the kinetic energy integral,

$$\frac{c^2}{2}[J(\dot{u}^2 + \dot{v}^2) + \dot{w}^2 \operatorname{ch}^2 v \sin^2 u] - \frac{fM}{c} \cdot \frac{\operatorname{sh} v - \delta \cos u}{\operatorname{sh}^2 v + \cos^2 u} = h \tag{6.42}$$

and the angular-momentum integral,

$$\dot{w} \operatorname{ch}^2 v \sin^2 u = c_1. \tag{6.43}$$

Excluding the cyclic coordinate w with the help of the integral (6.42), we represent the equations of motion in this form

$$\left.\begin{aligned}
&\frac{d}{dt}(J\dot{u}) + (\dot{u}^2 + \dot{v}^2)\sin u \cos u = \frac{\partial W}{\partial u},\\
&\frac{d}{dt}(J\dot{v}) - (\dot{u}^2 + \dot{v}^2)\operatorname{sh} v \operatorname{ch} v = \frac{\partial W}{\partial v},
\end{aligned}\right\} \tag{6.44}$$

where the altered force function is expressed by the formula /139

$$W = \frac{U}{c^2} - \frac{c_1^2}{2 \operatorname{ch}^2 v \sin^2 u}. \tag{6.45}$$

The kinetic energy integral is now transformed to

$$J(\dot{u}^2 + \dot{v}^2) = 2\left(W + \frac{h}{c^2}\right). \tag{6.46}$$

Just as in § 2 of the present chapter, we regularize the equations of motion (6.44) by introducing an independent variable τ, defined by this formula

$$dt = Jd\tau = (\mathrm{sh}^2 v + \cos^2 u)\, d\tau. \tag{6.47}$$

Transformations then enable us to arrive at the following differential equations:

$$\left.\begin{aligned} u'' - \frac{1}{2J}\frac{\partial J}{\partial u}(u'^2 + v'^2) &= J\frac{\partial W}{\partial u}, \\ v'' - \frac{1}{2J}\frac{\partial J}{\partial v}(u'^2 + v'^2) &= J\frac{\partial W}{\partial v}, \end{aligned}\right\} \tag{6.48}$$

where the prime marks denote differentiation with respect to τ. The system (6.48) admits of a kinetic energy integral,

$$u'^2 + v'^2 = 2J\left(W + \frac{h}{c^2}\right), \tag{6.49}$$

which in turn allows us to transform the equations of motion (6.48) to the following form:

$$\left.\begin{aligned} u'' &= J\frac{\partial W}{\partial u} + \frac{\partial J}{\partial u}\left(W + \frac{h}{c^2}\right), \\ v'' &= J\frac{\partial W}{\partial v} + \frac{\partial J}{\partial v}\left(W + \frac{h}{c^2}\right). \end{aligned}\right\} \tag{6.50}$$

The system (6.50) can be written as follows:

$$\left.\begin{aligned} u'' &= \frac{\partial \overline{W}}{\partial u}, \\ v'' &= \frac{\partial \overline{W}}{\partial v}, \end{aligned}\right\} \tag{6.51}$$

where the altered force function $\overline{W}$ is defined as

$$\overline{W} = J\left(W + \frac{h}{c^2}\right). \tag{6.52}$$

In explicit form the function $\overline{W}$ is defined as follows: /140

$$\overline{W} = (\mathrm{sh}^2 v + \cos^2 u)\left[\frac{fM}{c^3}\cdot\frac{\mathrm{sh}\, v - \delta\cos u}{\mathrm{sh}^2 v + \cos^2 u} - \frac{c_1^2}{2\,\mathrm{ch}^2 v \sin^2 u} + \frac{h}{c^2}\right], \tag{6.53}$$

and the kinetic energy integral assumes this form

$$u'^2 + v'^2 = 2\overline{W}. \tag{6.54}$$

Calculating the partial derivatives of the function $\overline{W}$ with respect to the variables u and v, we obtain

$$\left.\begin{aligned}\frac{\partial \overline{W}}{\partial u} &= \frac{fM\delta}{c^3}\sin u + \frac{c_1^2\cos u}{\sin^3 u} - \frac{2h}{c^2}\cos u\sin u,\\ \frac{\partial \overline{W}}{\partial v} &= \frac{fM}{c^3}\operatorname{ch} v - \frac{c_1^2\operatorname{sh} v}{\operatorname{ch}^3 v} + \frac{2h}{c^2}\operatorname{sh} v\operatorname{ch} v.\end{aligned}\right\} \tag{6.55}$$

As a result of the transformations, we have obtained a system of equations (6.48) which decomposes into two mutually independent equations.

$$u'' = \frac{fM\delta}{c^3}\sin u + \frac{c_1^2\cos u}{\sin^3 u} - \frac{2h}{c^2}\sin u\cos u, \tag{6.56}$$

$$v'' = \frac{fM}{c^3}\operatorname{ch} v - \frac{c_1^2\operatorname{sh} v}{\operatorname{ch}^3 v} + \frac{2h}{c^2}\operatorname{sh} v\operatorname{ch} v. \tag{6.57}$$

Multiplying equation (6.56) by u', and equation (6.57) by v', and integrating, we arrive at

$$\left(\frac{du}{d\tau}\right)^2 = \frac{2h}{c^2}\cos^2 u - \frac{2fM\delta}{c^3}\cos u - \frac{c_1^2}{\sin^2 u} + 2c_2', \tag{6.58}$$

$$\left(\frac{dv}{d\tau}\right)^2 = \frac{2h}{c^2}\operatorname{sh}^2 v + \frac{2fM}{c^3}\operatorname{sh} v + \frac{c_1^2}{\operatorname{ch}^2 v} + 2c_2. \tag{6.59}$$

Comparing the kinetic energy integral (6.54) with the first integrals of (6.58) and (6.59), we find that the constants c_2 and c'_2 are related as follows :

$$c_2 + c'_2 = 0. \tag{6.60}$$

Hereafter we shall retain only the constant c_2 in the equations.

Introducing the new variables /141

$$\lambda = \operatorname{sh} v, \qquad \mu = \cos u, \tag{6.61}$$

in place of the integrals (6.58) and (6.59), we obtain

$$\mu'^2 = -\frac{2h}{c^2}\mu^4 + \frac{2fM\delta}{c^3}\mu^3 + 2\left(c_2 + \frac{h}{c^2}\right)\mu^2 - \frac{2fM\delta}{c^3}\mu - (2c_2 + c_1^2),$$

$$\lambda'^2 = \frac{2h}{c^2}\lambda^4 + \frac{2fM}{c^3}\lambda^3 + 2\left(c_2 + \frac{h}{c^2}\right)\lambda^2 + \frac{2fM}{c^3}\lambda + (2c_2 + c_1^2),$$

from which, by integrating, we arrive at

$$\int \frac{d\mu}{\sqrt{f(\mu)}} = \tau + c_3, \tag{6.62}$$

$$\int \frac{d\lambda}{\sqrt{\Psi(\lambda)}} = \tau + c_4, \tag{6.63}$$

where

$$f(\mu) = -\frac{2h}{c^2}\mu^4 + \frac{2fM\delta}{c^3}\mu^3 + 2\left(c_2 + \frac{h}{c^2}\right)\mu^2 - \frac{2fM\delta}{c^3}\mu - 2c_2 - c_1^2,$$

$$\Psi(\lambda) = \frac{2h}{c^2}\lambda^4 + \frac{2fM}{c^3}\lambda^3 + 2\left(c_2 + \frac{h}{c^2}\right)\lambda^2 + \frac{2fM}{c^3}\lambda + 2c_2 + c_1^2.$$

Following inversion of the elliptical quadratures (6.62) and (6.63), the quantitites λ and μ are found as explicit functions of τ, and then the longitude w is determined by the quadrature

$$w = c_1 \int \frac{(\lambda^2 + \mu^2)\, d\tau}{(1 - \mu^2)(1 + \lambda^2)} + c_5. \tag{6.64}$$

Following this, we can find the time t from the formula

$$t = \int (\lambda^2 + \mu^2)\, d\tau. \tag{6.65}$$

Formulas (6.62) - (6.65) offer a general solution to the problem.

NOTE: The generalized problem of two immobile centers can be successfully applied in the study of artificial satellites of other planets than the earth. Calculations made by Ye. L. Lukashevich[1] have made it possible to obtain the following values for the parameters of approximating potentials of several planets: for Mars, c = 150,013 km, δ = 0; for Jupiter, c = 8,461.57 km, δ = 0; and for Saturn, c = 7,547.368 km, δ = 0. /142

The coefficient δ is everywhere taken equal to zero, since information on the numerical value of this coefficient for the third harmonic is completely lacking in the case of the large planets. To determine δ from observations of the natural satellites of the large planets is impossible, since it is associated with periodic rather than circular inequalities.

Only some general remarks can be made regarding the quality of the approximation. We do not know the coefficient J_4 for Mars. However, in view of the closeness of many of the physical characteristics of the earth and

[1]Ye. L. Lukashevich has kindly presented these figures to the author.

Mars, we may expect that the ratio J_4/J_2^2 will be of the same order for the two planets. The approximation of the Martian potential gives a value of 1.05 for J_4/J_2^2, but a corresponding value of 1 - 2 for the earth. J_4 for Jupiter is known to have a value of 0.000216 -- in other words, J_4 is of the same order for this planet. It should be noted that the effect of the fourth harmonic in the motion of the Galilean satellites is negligibly small. The known value of J_4 for Sature is 0.000278. Consequently, the approximating expression for the potential is satisfactory in this case also. We should note, in addition, that of the total annual angular motion of the Saturnian satellites Mimas and Tethys, amounting to 365°.23 and 72°.227, the effect of the fourth harmonic accounts for 5°.74 and 0°.32. From this we conclude that the inherent error of the approximation is insignificant.

§ 7. Approximation Solution of the Generalized Problem of Two Immobile Centers

In the case of remote artificial earth satellites, we can limit ourselves to an approximate solution of the generalized problem of two immobile centers [108]. We introduce a cylindrical system of coordinates $\tilde{r}$, ζ, λ, which is related to the system of rectangular geocentric coordinates as follows

$$\left.\begin{aligned} \tilde{r} &= \sqrt{x^2+y^2}, \\ \zeta &= z - c\delta, \\ \tan\lambda &= \frac{y}{x}. \end{aligned}\right\} \tag{7.1}$$

From formulas (3.10) and (3.12), Chapter I, we can easily derive the following Hamilton-Jacobi equation: /143

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial V}{\partial \tilde{r}}\right)^2 + \frac{1}{\tilde{r}^2}\left(\frac{\partial V}{\partial \lambda}\right)^2 + \left(\frac{\partial V}{\partial \zeta}\right)^2\right] - U = 0, \tag{7.2}$$

in which U is assumed to be independent of the longitude λ. From equation (7.2) we find that

$$V = -\alpha_1 t + \alpha_3 \lambda + W(\tilde{r}, \zeta), \tag{7.3}$$

where α_1 and α_3 are arbitrary constants; the constant W must satisfy the equation

$$\left(\frac{\partial W}{\partial \tilde{r}}\right)^2 + \left(\frac{\partial W}{\partial \zeta}\right)^2 - 2U - 2\alpha_1 + \frac{\alpha_3^2}{\tilde{r}^2} = 0. \tag{7.4}$$

Transforming equation (7.4) to oblate spheroidal coordinates, q_1, q_2, q_3, defined by the equations

$$q_1 = \frac{r_1 + r_2}{2}, \quad q_2 = \frac{r_2 - r_1}{2i}, \quad q_3 = \lambda \tag{7.5}$$

(formulas (8.15), Chapter I, enable us to find the connection between these spheroidal coordinates and those defined by formulas (3.20), Chapter I). From equations (7.5) we find

$$\left.\begin{aligned} \tilde{r} &= \frac{1}{c}\sqrt{(c^2 + q_1^2)(c^2 - q_2^2)}, \\ \rho &= \sqrt{x^2 + y^2 + (z - c\delta)^2} = \sqrt{q_1^2 - q_2^2 + c^2}, \\ \zeta &= \frac{q_1 q_2}{c^2}. \end{aligned}\right\} \tag{7.6}$$

In spheroidal coordinates q_i, the Hamilton-Jacob equation (7.4) has the form

$$\begin{aligned} (q_1^2 + c^2)\left(\frac{\partial W}{\partial q_1}\right)^2 + (c^2 - q_2^2)\left(\frac{\partial W}{\partial q_2}\right)^2 - 2fM(q_1 - \delta q_2) - \\ - 2\alpha_1(q_1^2 + q_2^2) + \frac{\alpha_3^2 c^2}{c^2 - q_2^2} - \frac{\alpha_3^2 c^2}{c^2 + q_1^2} = 0. \end{aligned} \tag{7.7}$$

In equation (7.7) in place of U there now appears an explicit expression for the potential of the generalized problem of two immobile centers (6.41). /144
From equations (7.3) and (7.7), it is easy to derive the total integral of the Hamilton-Jacobi equation:

$$\begin{aligned} V = &-\alpha_1 t + \alpha_3 q_3 + \\ &+ \int \sqrt{(q_1^2 + c^2)(2\alpha_1 q_1^2 + 2fMq_1 + \alpha_2) + \alpha_3^2 c^2}\,\frac{dq_1}{q_1^2 + c^2} + \\ &+ \int \sqrt{(c^2 - q_2^2)(2\alpha_1 q_2^2 - 2fM\delta q_2 - \alpha_2) - \alpha_3^2 c^2}\,\frac{dq_2}{c^2 - q_2^2} \end{aligned} \tag{7.8}$$

(See (8.13), Chapter I).

The total integral (7.8) enables us to find the general integral of the problem in terms of the variables q_i. We can obtain an approximate solution by making use of the expressions of (8.18), Chapter I. Limiting ourselves to the first degree of the small quantity c, we obtain

$$\left.\begin{aligned} r_1 &\cong \rho\left(1-\frac{c\zeta i}{\rho^2}\right), \\ r_2 &\cong \rho\left(1+\frac{c\zeta i}{\rho^2}\right), \end{aligned}\right\} \tag{7.9}$$

from which we find

$$q_1 \approx \rho, \tag{7.10}$$

$$q_2 \approx \frac{c\zeta}{\rho} = c\sin\psi, \tag{7.11}$$

where

$$\sin\psi = \frac{\zeta}{\rho}.$$

Then in place of (7.8) we obtain

$$V = -\alpha_1 t + \alpha_3\lambda + \int_{\rho_0}^{\rho} \sqrt{2\alpha_1\rho^2 + 2fM\rho + \alpha_2}\,\frac{d\rho}{\rho} + \int_0^{\psi} \sqrt{\cos^2\psi\,(2fMc\delta\sin\psi - \alpha_2) - \alpha_3^2}\,\frac{d\psi}{\cos\psi}, \tag{7.12}$$

where r_0 is a certain constant.

From (7.12) in the usual manner (see (4.19), Chapter I) we find the general integral: /145

$$\int_{\rho_0}^{\rho} \frac{\rho\,d\rho}{\sqrt{2\alpha_1\rho^2 + 2fM\rho + \alpha_2}} = t + \beta_1, \tag{7.13}$$

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\rho\sqrt{2\alpha_1\rho^2 + 2fM\rho + \alpha_2}} - \int_0^{\psi} \frac{\cos\psi\,d\psi}{\sqrt{-2fMc\delta\sin^3\psi + \alpha_2\sin^2\psi + 2fMc\delta\sin\psi - \alpha_3^2 - \alpha_2}} = \beta_2, \tag{7.14}$$

$$\lambda = \beta_3 + \alpha_3\int_0^{\psi} \frac{d\psi}{\cos\psi\sqrt{-2fMc\delta\sin^3\psi + \alpha_2\sin^2\psi + 2fMc\delta\sin\psi - \alpha_3^2 - \alpha_2}}, \tag{7.15}$$

where β_i designates the canonical constants.

The canonical constants α_i and β_i are unsuitable for our purpose, since they do not afford a sufficiently simple mechanical and geometrical interpretation. Therefore, it is preferable to introduce another system of elements, which, with $c = \delta = 0$, will coincide with the system of elements characterizing Keplerian motion (with $c = \delta = 0$, formulas (7.13) - (7.15) define Keplerian motion, as follows from the potential (6.41)).

Let us suppose that the motion takes place within a limited portion of space in the vicinity of the gravitating body. This occurs when the constant of kinetic energy α_1 is negative. In this case, the motion may be called quasi-elliptical.

Let us denote the roots of the quadratic equation

$$2\alpha_1\rho^2 + 2fM\rho + \alpha_2 = 0$$

with the symbols ρ_1 and ρ_2. Both roots are positive, provided

$$\alpha_1 < 0, \qquad \alpha_2 < 0.$$

setting

$$\rho_1 = a(1 - e), \quad \rho_2 = a(1 + e),$$

the canonical elements α_1, α_2 will be associated with the elements a and e as follows: /146

$$\alpha_1 = -\frac{fM}{2a}, \tag{7.16}$$

$$\alpha_2 = -fMp, \tag{7.17}$$

where $p = a(1 - e^2)$. The quantities a, e, p we shall refer to as the major semi-axis, the eccentricity and the parameter of the quasi-elliptical.

Let

$$\rho_0 = a(1 - e),$$

and let $\rho = \rho_0$ if $t = T$. Then from (7.13) we find that

$$\beta_1 = -T \tag{7.18}$$

The quantity T is the moment of passage through the perigee (see (7.17), Chapter I).

Analysis of the equation

$$-2fMc\delta \sin^3\psi + \alpha_2 \sin^2\psi + 2fMc \sin\psi - \alpha_3^2 - \alpha_2 = 0 \tag{7.19}$$

shows that it is advantageous to substitute, in place of the constant α_3, the

element i, which is defined as follows:

$$\alpha_3 = \cos i \cdot \sqrt{fM(p + 2c\delta \sin i)}. \tag{7.20}$$

In place of the constant β_2 we introduce the quantity ω:

$$\beta_2 = -\frac{\omega}{\sqrt{fMp}}. \tag{7.21}$$

If $c = \delta = 0$, this particular quantity coincides with the distance between the pericenter and the node, while the angle i measures the inclination of the orbit.

Finally, in place of β_3 we introduce the quantity Ω

$$\beta_3 = \Omega, \tag{7.22}$$

which represents the longitude of the ascending node (see (7.26), Chapter I).

Let us examine equation (7.13). We set

$$\rho = a(1 - e\cos E). \tag{7.23}$$

Then, following integration, in place of equation (7.13) we have /147

$$E - e \sin E = n(t - T) \tag{7.24}$$

where

$$n^2 = \frac{fM}{a^3}. \tag{7.25}$$

In addition, let

$$\rho = \frac{p}{1 + e\cos v}. \tag{7.26}$$

From (7.23) and (7.26) we find that

$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}. \tag{7.27}$$

The first of the integrals of equation (7.14) is calculated by means of the substitution of (7.26)

$$\int_{a(1-e)}^{\rho} \frac{d\rho}{\rho \sqrt{2\alpha_1\rho^2 + 2fM\rho + \alpha_2^2}} = \frac{v}{\sqrt{fMp}}. \tag{7.28}$$

Let us examine the second integral which appears in (7.14):

$$I = \int_0^{\psi} \frac{\cos\psi\, d\psi}{\sqrt{-2fMc\delta \sin^3\psi + \alpha_2 \sin^2\psi + 2fMc\delta \sin\psi - \alpha_2 - \alpha_3^2}}. \tag{7.29}$$

Setting

$$\xi = \sin\psi, \tag{7.30}$$

we find that

$$I = \int_0^{\xi} \frac{d\xi}{\sqrt{2fMc\delta\,(\xi_1 - \xi)\,(\xi - \xi_2)\,(\xi - \xi_3)}},$$

where ξ_i represents the roots of equation (7.19):

$$\left.\begin{aligned} \xi_1 &= \sin i, \\ \xi_{2,3} &= \frac{1}{4c\delta}\left[-p - 2c\delta \sin i \pm \sqrt{(p - 2c\delta \sin i)^2 + 16c^2\delta^2 \cos^2 i}\right]. \end{aligned}\right\} \tag{7.31}$$

The approximated values of the roots ξ_2 and ξ_3 will be /148

$$\xi_2 \approx -\sin i, \qquad \xi_3 \approx -\frac{p}{2c\delta}.$$

With the help of the substitution

$$\xi = (\xi_2 - \xi_1)\sin^2\tilde{\varphi} + \xi_1 \tag{7.32}$$

the integral (7.29) reduces to the following form:

$$I = \sqrt{\frac{2}{fMc\delta\,(\xi_1 - \xi_3)}}\, F(\tilde{\varphi}, k), \tag{7.33}$$

where $F(\phi, k)$ is an elliptical integral of the first type, while its module k satisfies the equation

$$k^2 = \frac{\xi_1 - \xi_2}{\xi_1 - \xi_3}. \tag{7.34}$$

Then from (7.14) we find that

$$u = \mathrm{F}(\widetilde{\varphi}, k), \tag{7.35}$$

while

$$u = \sqrt{\frac{c\delta(\xi_1 - \xi_3)}{2p}}(v + \omega). \tag{7.36}$$

From (7.35) we obtain

$$\widetilde{\varphi} = \mathrm{am}(u, k). \tag{7.37}$$

From formulas (7.30), (7.32) and (7.37) it follows that

$$\sin\psi = \xi_1 + (\xi_2 - \xi_1)\,\mathrm{sn}^2 u. \tag{7.38}$$

Let us turn now to the integral under consideration:

$$I_1 = \int_0^{\psi} \frac{d\psi}{\cos\psi \sqrt{-2fMc\delta\sin^3\psi + \alpha_2\sin^2\psi + 2fMc\delta\sin\psi - \alpha_2 - \alpha_3^2}},$$

which forms part of equation (7.15). By means of substituting (7.32) integral I_1 is transformed to normal form:

$$I_1 = \frac{1}{\sqrt{2fMc\delta(\xi_1 - \xi_3)}}\left[\frac{1}{1-\xi_1}\int_0^{\widetilde{\varphi}} \frac{d\widetilde{\varphi}}{(1 + n'\sin^2\widetilde{\varphi})\sqrt{1 - k^2\sin^2\widetilde{\varphi}}} + \right.$$
$$\left. + \frac{1}{1+\xi_1}\int_0^{\widetilde{\varphi}} \frac{d\widetilde{\varphi}}{(1 + n''\sin^2\widetilde{\varphi})\sqrt{1 - k^2\sin^2\widetilde{\varphi}}}\right], \tag{7.39}$$

where the parameters of the integrals of third type in (7.39) are /149

$$n' = \frac{\xi_1 - \xi_2}{1 - \xi_1}, \quad n'' = \frac{\xi_2 - \xi_1}{1 + \xi_1}. \tag{7.40}$$

From (7.15) and (7.39) it is easy to obtain an equation which determines the right ascension (geographical longitude) of the satellite:

$$\lambda - \Omega_0 = \cos i \cdot \sqrt{\frac{p + 2c\delta\sin i}{2c\delta(\xi_1 - \xi_3)}}\left[\frac{1}{1-\xi_1}\Pi(\mathrm{am}\,u, n', k) + \frac{1}{1+\xi_1}\Pi(\mathrm{am}\,u, n'', k)\right]. \tag{7.41}$$

The formulas obtained completely define the motion of an artificial earth satellite.

The approximate solution of the generalized problem of two immobile centers has the same structure as that of the solution of Barrar's problem. This identity in structure of the solution is of course not accidental. It is due to the fact that in the expansions in series we obtained the Barrar potential from the potential of the problem of two immobile centers.

§ 8. The Limiting Variant of the Problem of Two Immobile Centers and Its Application in Celestial Ballistics

In a rectangular planetocentric system of coordinates whose x-axis is directed toward the perturbing body (Figure 20), the gravitational potential of the two immobile centers is written as follows:

$$U = \frac{fm}{r} + \frac{fM}{\sqrt{r^2 - 2cx + c^2}}, \tag{8.1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ while c is the distance between centers.

Assuming that the planetocentric distance r is small in comparison with c, we expand potential (8.1) in series in Legendre polynomials:

$$U = \frac{fm}{r} + \frac{fM}{c}\sum_{k=0}^{\infty}\left(\frac{r}{c}\right)^k P_k\left(\frac{x}{r}\right). \tag{8.2}$$

As was done by Hill [110], we limit ourselves to the first terms of the expansion, i.e., we consider the problem of two immobile centers without making allowance for the "parallax" of the perturbing body: /150

$$U = \frac{fm}{r} + \frac{fM}{c^2}x. \tag{8.3}$$

The force field under consideration consists of superimposed fields: Newtonian gravitational field of a point mass, a uniform field.

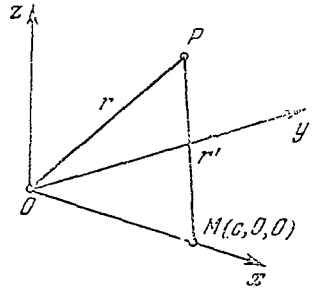

Figure 20.

This problem is integrated in paraboloidal coordinates (see § 3, Chapter I):

$$\left.\begin{aligned} q_1 &= \frac{1}{2}(r + x),\\ q_2 &= \frac{1}{2}(r - x),\\ q_3 &= \operatorname{arctg}\frac{y}{z}. \end{aligned}\right\} \tag{8.4}$$

Here we have introduced other paraboloidal coordinates than those which appear in (3.39) Chapter I; the latter are associated with the coordinates introduced above by the following relationships:

$$\xi\eta = 2\sqrt{q_1 q_2}, \quad \xi^2 - \eta^2 = 2(q_1 - q_2), \quad q_3 + \varphi = \frac{\pi}{2}.$$

Making the corresponding changes in formula (3.40) Chapter I, we obtain the following expression for kinetic energy:

$$T = \frac{1}{2}\left[(q_1 + q_2)\left(\frac{\dot{q}_1^2}{q_1} + \frac{\dot{q}_2^2}{q_2}\right) + 4\dot{q}_3^2\right], \tag{8.5}$$

while the force function will be as follows:

$$U = \frac{fm}{q_1 + q_2} + \frac{fM}{c^2}(q_1 - q_2). \tag{8.6}$$

Then the Hamilton-Jacobi equation (8.27) Chapter I will be as follows:

$$q_1\left(\frac{\partial V}{\partial q_1}\right)^2 + q_2\left(\frac{\partial V}{\partial q_2}\right)^2 + \frac{q_1 + q_2}{4q_1 q_2}\alpha_3^2 - 2(q_1 + q_2)\alpha_1 - 2fm\left[1 + \frac{M}{mc^2}(q_1^2 - q_2^2)\right] = 0. \tag{8.7}$$

In correspondence with equation (8.27) Chapter I the total integral is determined by formula:

$$V = -\alpha_1 t + \alpha_3 q_3 + \int \sqrt{2Q(q_1)}\,\frac{dq_1}{q_1} + \int \sqrt{2P(q_2)}\,\frac{dq_2}{q_2}, \tag{8.8}$$

where

$$\left.\begin{aligned} Q(q_1) &= \frac{fM}{c^2}q_1^3 + \alpha_1 q_1^2 + \frac{1}{2}\alpha_2 q_1 - \frac{1}{8}\alpha_3^2, \\ P(q_2) &= -\frac{fM}{c^2}q_2^3 + \alpha_1 q_2^2 - \frac{1}{2}(\alpha_2 - fm)q_2 - \frac{1}{8}\alpha_3^2, \end{aligned}\right\} \tag{8.9}$$

while α_i represents the canonical constants. From (8.8) we find the general integral of the problem:

$$\int \frac{q_1 dq_1}{\sqrt{2Q(q_1)}} + \int \frac{q_2 dq_2}{\sqrt{2P(q_2)}} = t - \beta_1, \tag{8.10}$$

$$\int \frac{dq_1}{\sqrt{2Q(q_1)}} - \int \frac{dq_2}{\sqrt{2P(q_2)}} = -2\beta_2, \tag{8.11}$$

$$\frac{\alpha_3}{4}\left[\int\frac{dq_1}{q_1\sqrt{2Q(q_1)}}+\int\frac{dq_2}{q_2\sqrt{2P(q_2)}}\right]=q_3+\beta_3. \tag{8.12}$$

The presentation of the motion of a particle in a Newtonian central field in the presence of a constant perturbing force is described in an article by the present writer [111]. V. V. Beletskiy [16] has approached the problem differently, considering the motion of particles in a central Newtonian field of force in the presence of a force vector of reactive acceleration. Careful and refined qualitative study of the forms of motion in the three dimensional problem has been made by A. L. Kunitsyn [17, 18], for the purpose of determining the perturbed effect of solar light pressure [112]. Other applications of this problem in the field of celestial ballistics may be found in [113, 114].

In the construction of approximate solutions to the problems of astrodynamics, frequent use is made of the method of the gravitational spheres of celestial bodies [115]. Spheres of influence, the Hill gravitational sphere, ordinary gravitational spheres, etc., are analyzed. Proceeding from the limiting circular 3 body problem, M. D. Kislik [116] introduced the concept of the "sphere of influence", the radius of which is approximately 50% greater than the radius of the Laplascian sphere of influence. It would seem advantageous to introduce a gravitational sphere based on the limiting analyzed variant of the problem of two immobile center, which is studied here.

Let us now proceed to a brief examination of the results obtained by A. L. Kunitsyn. Most interesting as regards applications is the class of motions with negative energy $\alpha_1 < 0$. The character of orbital motion depends upon the values of the constants α_i. Let the roots of the equations /152

$$P(q_2)=0, \tag{8.13}$$

$$Q(q_1)=0 \tag{8.14}$$

be κ, λ, ν and α, β, γ, respectively. Let us determine the boundaries of the regions of possible motion, as illustrated in Figure 21. Of the two possible types of motion illustrated in Figure 21 by the shaded areas, let us consider the one which occurs in the toroidal region formed by the intersection of the four paraboloids. The trajectories of this type always lie within a limited portion of space, and exhibit a satellite character.

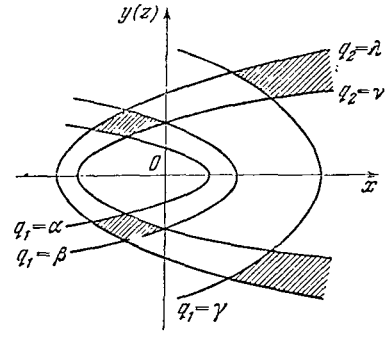

Figure 21.

Just as in § 2, we now introduce an independent variable τ

$$(q_1+q_2)\,d\tau=dt. \tag{8.15}$$

Then equations (8.10) and (8.11) will yield the following

$$\int_{q_{10}}^{q_1} \frac{dq_1}{\sqrt{2(q_1-\gamma)(q_1-\alpha)(q_1-\beta)}} = \sqrt{\frac{fM}{c^2}}(\tau-\tau_0), \tag{8.16}$$

$$\int_{q_{20}}^{q_2} \frac{dq_2}{\sqrt{2(q_2-\varkappa)(q_2-\lambda)(\nu-q_2)}} = \sqrt{\frac{fM}{c^2}}(\tau-\tau_0) \tag{8.17}$$

(the following initial conditions are assumed: for $\tau = \tau_0$, the equality $q_1 = q_{10}$, $q_2 = q_{20}$ is fulfilled).

Then from (8,12) we find

$$q_3 = \frac{\alpha_3}{4}\int \frac{(q_1+q_2)\,d\tau}{q_1 q_2}. \tag{8.18}$$

We reduce the elliptical quadrature (8.17) to normal Legendre form with the help of the substitution /153

$$q_2 = \nu - (\nu-\lambda)\sin^2\varphi \tag{8.19}$$

and, inverting integral obtained, we arrive at

$$q_2 = \lambda + (\nu-\lambda)\,\mathrm{cn}^2\,[(u_2+u_{20}), k_2], \tag{8.20}$$

where

$$u_2 = m_2(\tau-\tau_0), \qquad m_2 = \sqrt{\frac{fM(\nu-\varkappa)}{2c^2}}, \tag{8.21}$$

while the module of the elliptical integral k_2 defined by the equality

$$k_2^2 = \frac{\nu-\lambda}{\nu-\varkappa}. \tag{8.22}$$

In addition, we substitute

$$u_{20} = F\left(\arcsin\sqrt{\frac{\nu-q_{20}}{\nu-\lambda}},\ k_2\right) \tag{8.23}$$

For transformation of the elliptical integral (8.16) we make use of the substitution

$$q_1 = \alpha + (\beta-\alpha)\sin^2\varphi \tag{8.24}$$

and, in addition, inverting the elliptical integral, we arrive at the formula

$$q_1 = \alpha + (\beta - \alpha)\,\mathrm{sn}^2\,[(u_1 + u_{10}), k_1], \tag{8.25}$$

where

$$u_1 = m_1(\tau - \tau_0), \quad m_1 = \sqrt{\frac{fm(\gamma - \alpha)}{2c^2}}, \tag{8.26}$$

$$u_{10} = \mathrm{F}\left(\arcsin\sqrt{\frac{q_{10} - \alpha}{\beta - \alpha}}, k_1\right). \tag{8.27}$$

The quadrature (8.18), following the appropriate transformations, assumes the following form:

$$q_3 = q_{30} + \frac{1}{4}\alpha_3\left\{\frac{1}{\alpha m_1}\Pi\,[\mathrm{am}\,(u_1 + u_{10}), k_1, n_1] + \frac{1}{\nu m_2}\Pi\,[\mathrm{am}\,(u_2 + u_{20}), k_2, n_2]\right\}, \tag{8.28}$$

where

$$n_1 = \frac{\beta - \alpha}{\alpha}, \quad n_2 = \frac{\lambda - \nu}{\nu}.$$

Finally, the equation of time is obtained from (8.15) in the following form: /154

$$\begin{aligned} t - t_0 = {} & (\varkappa - \gamma + 2\beta)(\tau - \tau_0) + \\ & + \frac{\gamma - \alpha}{m_1}\{\mathrm{E}\,[\mathrm{am}\,(u_1 + u_{10})] - \mathrm{E}\,(\mathrm{am}\,u_{10})\} + \\ & + \frac{\nu - \varkappa}{m_2}\{\mathrm{E}\,[\mathrm{am}\,(u_2 + u_{20})] - \mathrm{E}\,(\mathrm{am}\,u_{20})\}. \end{aligned} \tag{8.29}$$

In conclusion let us apply this problem to estimating the perturbations of light pressure in the case of the "Echo-I"[1] satellite, which consisted of an inflated balloon for which the ratio of the cross-sectional area to the mass of the satellite, S/M_c, was approximately 100 $m^3 \cdot sec^2/kg$. The investigations described in [117] demonstrated that in the course of 12 hours of flight the perigee distance of the Echo-1 satellite was reduced by 42 km under the effect of light pressure. To utilize the results obtained in the solution, the quanity $f\,M/c^2$ should be replaced with the perturbing acceleration of light pressure μ, which is defined as

$$\mu = kp_0\frac{S}{M_c}\left(\frac{a}{r_c}\right)^2,$$

[1] Results of the calculations were kindly forwarded to me by A. L. Kunitsyn.

where k is a coefficient which depends upon the albedo of the satellite, p_0 is the pressure of luminous solar radiation at the earth's orbit for a black body ($p_0 = 4.5 \cdot 10^{-7}$ kg/m^2), a is the major semi-axis of the earth's orbit, and r_c is the distance from the satellite to the sun. From this, for k = 1, we find that $\mu = 4.5 \cdot 10^{-5}$ m/sec^2.

Assuming that r = 7,970 km, and performing the calculations, we find the relationship between perturbation of the radius-vector and the polar angle. We discover that even during a single loop by the satellite, perturbation of the radius vector may reach a value of approximately 0.6 km. Use of the orbits of the limiting variant of the problem of two immobile centers in place of intermediate orbits does not lead to any secular perturbation in the radius vector. At the same time, use of the classical method of calculating the perturbations inevitably leads to the presence of secular terms. This is the basis of the superiority of the limiting variant of the problem of two immobile centers over the two-body problem in the present situation.

§ 9. Qualitative Estimates of Approximating Potentials

The smallness of the difference between the true and the approximating potential does not, strictly speaking, afford a sufficient basis for presuming deviations in the coordinates. Thus, our earlier assertions regarding the quality of the approximation are based on intuitive rather than on mathematical considerations. Comparison of the analytical theory with the results of /155
numerical integration of the equations of motion leads to more or less exact conclusions regarding individual trajectories, But it does not afford any evaluation of the approximation as a whole. The problem becomes particularly complicated when the longitudinal terms in the gravitational potential are taken into account.

Nevertheless, it is quite possible to formulate some idea of the quality of the approximation, at least as regards individual coordinates, by proceeding on the basis of qualitative methods of analysis. Of particular value in this connection is Hill's method, which is based on analysis of the curve (or surface) of zero velocity, and the method of contact characteristics developed by Hadamard [118] and Poincaré [119].

Hill's method enables us to determine the stability of motion in the Lagrangian sense, using the kinetic energy integral (or the Jacobi integral). This method has been applied repeatedly in the plane secural three-body problem (G. Hill [120], Bohlin [121], Darwin [122], N. D. Moiseyev [123] and others.)

If the equations of motion admit of a kinetic energy integral

$$v^2 = 2[U(x, y, z) + h], \tag{9.1}$$

where U is the force function, v is the velocity of the point, then it is necessary to consider the surface

$$U(x, y, z) + h = 0 \tag{9.2}$$

which is called the surface of zero velocity (or Hill's surface). Since motion is possible only if

$$U + h \geq 0,$$

the surface (9.2) breaks down spatially into a region of possible motion ($U + h \geq 0$) and a region in which motion is impossible ($U + h < 0$). If the region of possible motion is closed, then motion will always take place in this region, and the trajectory will be stable in the Lagrangian sense.

For our purposes the problem can be formulated somewhat differently: we investigate Lagrangian stability in the presence of continuously active conservative perturbations -- in other words, we study the form of the surface of zero velocity for a small variation in the force function. /156

In the problem of the motion of the particle in an axisymmetric field of forces, Hill's method reduces to the following. In cylindrical coordinates the equations of motion have this form:

$$\left.\begin{aligned} \ddot{\rho} - \rho\dot{\lambda}^2 &= U'_\rho, \\ \frac{d}{dt}(\rho^2\dot{\lambda}) &= 0, \\ \ddot{z} &= U'_z. \end{aligned}\right\} \tag{9.3}$$

For simplification of the procedure and for obtaining more precise evaluations, we lower the degree of (9.3) with the help of the angular-momentum integral,

$$\rho^2\dot{\lambda} = c. \tag{9.4}$$

As a result we arrive at the following system

$$\left.\begin{aligned} \ddot{\rho} &= W'_\rho, \\ \ddot{z} &= W'_z, \end{aligned}\right\} \tag{9.5}$$

in which

$$W = U - \frac{c^2}{2\rho^2}, \tag{9.6}$$

while the kinetic energy integral becomes

$$v^2 = 2(W + h). \tag{9.7}$$

On the surface (ρ, z) this integral defines the curve of zero velocity,

$$W(\rho,z) + h = 0. \tag{9.8}$$

Substituting in W, in place of U, the approximating or the true potential, we arrive at

$$\frac{fM}{r}\left[1 + \sum_{k=2}^{\infty} J_k \left(\frac{R}{r}\right)^k P_k\left(\frac{z}{r}\right)\right] - \frac{c^2}{2\rho^2} + h = 0 \tag{9.9}$$

or

$$\frac{fM}{r}\left[1 + \sum_{k=2}^{\infty} J_k \left(\frac{R}{r}\right)^k P_k(\sin\varphi)\right] - \frac{c^2}{2r^2\cos^2\varphi} + h = 0 \tag{9.10}$$

(here $\sin\phi = z/r$).

For purposes of illustration, let us consider the case in which $J_k = 0$ in every instance -- in other words, the two-body problem. Then equation (9.10) assumes the form /157

$$\frac{fM}{r} - \frac{c^2}{2r^2\cos^2\varphi} + h = 0, \tag{9.11}$$

whence

$$r = \frac{-fM\cos\varphi \pm \sqrt{f^2M^2\cos^2\varphi + 2hc^2}}{2h\cos\varphi}. \tag{9.12}$$

If $h < 0$, then, substituting

$$\cos i = \frac{c\sqrt{-2h}}{fM}, \tag{9.13}$$

instead of (9.12) we will have

$$r = -\frac{fM}{h}(1 \mp \sqrt{1 - \cos^2 i \sec^2\varphi}). \tag{9.14}$$

If, on the other hand, $J_k \neq 0$, then (9.10) yields

$$r = -\frac{fM}{2h}\left[1 + \Phi \pm \sqrt{(1+\Phi)^2 + \frac{2hc^2}{f^2M^2\cos^2\varphi}}\right], \tag{9.15}$$

where

$$\Phi = \sum_{k=2}^{\infty} J_k \left(\frac{R}{r}\right)^k P_k(\sin\varphi). \tag{9.16}$$

Assigning to J_k the values which are of interest to us, we arrive at the region of possible motion, and on this basis we can guess at certain properties of the motion.

Let us proceed not to the second method of qualitative study -- that proposed by Poincaré [119] for a system of two differential equations of the first degree, and based on consideration of the mutual behavior of the integral curves and the curves of the uniparametric ("topographic") family,

$$f(x, y) = d. \tag{9.17}$$

Hadamard [118] has applied this method to dynamic systems of the form

$$\ddot{x} = U'_x, \quad \ddot{y} = U'_y. \tag{9.18}$$

N. D. Moiseyev has generalized it to cover the case of irreversible dynamic problems [123].

Let us consider first the system (9.18), assuming the existence of a kinetic energy integral, /158

$$\dot{x}^2 + \dot{y}^2 = 2(U + h). \tag{9.19}$$

We shall also assume that equation (9.17) defines the quanity d as a positive, single-valued, continuous function of x, y. Suppose we have a certain isoenergetic family of trajectories of system (9.18) -- i.e., a family of trajectories with fixed values of h. Among the integral curves passing through the point P (x_0, y_0) there will be found at least two which at point P are tangent to the curve

$$f(x, y) = f(x_0, y_0). \tag{9.20}$$

The condition of contacts of the integral and the topographic curves (9.20) is

formulated as follows:

$$\dot{f} = f'_x \dot{x} + f'_y \dot{y} = 0. \tag{9.21}$$

If the differential element of the orbit lies on the whole within the curve (9.20), then internal contact is present. If it lies outside curve (9.20), however, we refer to "external" contact. Finally, if the differential elements of the two curves coincide, then the contact is of higher order. The character of the contact depends upon the sign of $\ddot{f}$: for internal contact, $\ddot{f} < 0$ and for external contact $\ddot{f} > 0$. For $\ddot{f}$, with allowance for the kinetic energy integral we can obtain the following expression:

$$\ddot{f} = [2(U+h)(f''_{xx}f'^2_y - 2f''_{xy}f'_x f'_y + f''_{yy}f'^2_x) + (f'^2_x + f'^2_y)(f'_x U'_x + f'_y U'_y)]. \tag{9.22}$$

It is obvious that the curve

$$\ddot{f} = 0 \tag{9.23}$$

represents the boundary separating the region of external contacts from the region of internal contacts.

Equations similar to (9.5) will be found convenient in the study of satellites if we take a system of concentric circles as the family of topographic curves:

$$f(\rho, z) = \rho^2 + z^2 = r^2 = d^2. \tag{9.24}$$

If a family of ellipses is chosen, the use of elliptical coordinates will simplify the procedure. /159

For example, let us take the topographic family of circles (9.24) on the plane (ρ, z) (here, actually, we have a topographic family of spheres). Since /163

$$f'_\rho = 2\rho, \quad f'_z = 2z, \quad f''_{\rho\rho} = f''_{zz} = 2, \quad f''_{\rho z} = 0,$$

then for the characteristic of contacts we will have

$$\ddot{f} = 4(U+h) + 2(\rho U'_\rho + z U'_z). \tag{9.25}$$

This curve will divide the region of internal contacts from the region of external contacts. With its help we can divide the region of pericenters from the region of apocenters. It would evidently be useful to study the characteristic of contacts in the problem of the motion of a satellite moving in the equatorial plane of a triaxial planet.

CHAPTER IV

QUALITATIVE ANALYSIS OF THE FORMS OF MOTION OF A SATELLITE OF A SPHEROIDAL PLANET

§ 1. Motions of Elliptical Type /160

Qualitative analysis of possible forms of motion may be found in the works of Ye. P. Aksenov, Ye. A. Grebenikov and V. G. Demin [86, 87, 88, 89], for the case in which $\delta = 0$, i.e., without allowance for the asymmetry of the earth with respect to the equatorial plane. In the general case, the types of motion have been analyzed by Yu. I. Ivanov [124]. Fam Van Chi [125] has studied one class of motions. The geneology of solutions has been carefully studied by V. M. Alekseyev [31].

We shall restrict ourselves here to a study of the case $\delta = 0$. Here any form of motion can be assigned to one of three classes, depending on the sign of the constant h, which is the total mechanical energy. Motions for $h < 0$ we shall call elliptical; motions for $h > 0$ we shall call hyperbolic; and motions for $h = 0$, we shall call parabolic.

This terminology corresponds to the types of motion in the two-body problem. The terminology is convenient since, for $c = \delta = 0$, the equations of motion become differential equations with respect to the motion of the two-body problem. In this case motions assigned to the first class will follow Keplerian ellipses; those of the second class, hyperbolas; and those of the third class, parabolas.

Certain of the trajectories of this problem lie entirely within the body of the earth itself. These are of no practical interest and we have therefore excluded them from the present discussion. Trajectories which lie at least partially above the surface of the earth we shall refer to as "real". If a trajectory lies entirely above the surface of the earth and at least partially in outer space, we shall refer to it as "of satellite type". Those trajectories which lie partially within the body of the earth we shall refer as "ballistic". /161

For convenience we shall introduce the designation $-h_1c^2 = h$. Then, with due allowance for the observations made above, in place of formulas (6.62) and (6.63), Chapter 3, we will have the following

$$\mu'^2 = 2h_1\mu^4 + 2(c_2 - h_1)\mu^2 - (2c_2 + c_1^2), \tag{1.1}$$

$$\lambda'^2 = -2h_1\lambda^4 - \frac{2fM}{c^3}\lambda^3 + 2(c_2 - h_1)\lambda^2 - \frac{2fM}{c^3}\lambda + 2c_2 + c_1^2, \tag{1.2}$$

$$w'^2 = \frac{c_1(\lambda^2 + \mu^2)}{(1 - \mu^2)(\lambda^2 + 1)}, \tag{1.3}$$

where $\lambda = -\text{sh } v$, $\mu = \cos u$.

1. Study of the coordinate μ. The character of the solution of equation (1.1) for coordinate μ depends upon the roots of the polynomial

$$f(\mu) = 2h_1\mu^4 + 2(c_2 - h_1)\mu^2 - (2c_2 + c_1^2), \tag{1.4}$$

which are defined by the formula

$$\mu = \pm\sqrt{\frac{h_1 - c_2 \pm \sqrt{(h_1 + c_2)^2 + 2h_1c_1^2}}{2h_1}}. \tag{1.5}$$

We shall denote these roots by the symbols $\pm\mu_1$ and $\pm\mu_2$. The discriminant

$$\Delta = (h_1 + c_2)^2 + 2h_1c_1^2$$

with $h_1 > 0$ is always negative. Therefore, of the four roots of the polynomial $f(\mu)$ at least two (in order to be specific we have denoted them by symbol $\pm\mu_1$) will be real, and

$$|\mu_1| \geq 1. \tag{1.6}$$

Depending upon the magnitude of the other roots $\pm\mu_2$, a number of different cases may occur:

a) roots $\pm\mu_2$ are complex;

b) roots $\pm\mu_2$ are real but are greater than unity in absolute value;

c) roots $\pm\mu_2$ are real but in absolute value are less than unity;

d) $\mu_2 = 0$;

e) roots $\pm\mu_2$ are real, and $|\mu_2| = 1$;

f) $|\mu_1| = |\mu_2| = 1$.

From equations (1.3) and (1.4) it is evident that the region of permissible/162 values of μ corresponding to real motion is defined by the inequality

$$f(\mu) \geq 0. \tag{1.7}$$

In case a) the polynomial $f(\mu)$ satisfies condition (1.7) and $|\mu| \geq |\mu_1| > 1$,

which is impossible, since by reason of (6.61) Chapter 3, $|\mu| \leq 1$. Therefore case a) is meaningless. Case b) also turns out to be meaningless, since the inequality $|\mu_2| > 1$ leads to the condition $2h_1 c_1^2 < 0$, which contradicts the assumption that the orbit belongs to the elliptical class. In case d), $z = 0$, i.e., the motion takes place in the equatorial plane. Case e) occurs when $c_1 = 0$. But then, from (1.3) it follows that w remains constant, and the motion takes place in the meridional plane. This particular case will be discussed in detail in one of the subsequent paragraphs. As regards case d), there is no need for discussion within the framework of the generalized problem of two immobile centers, since the exact equations of motion of an artificial earth satellite in the equatorial plane can be integrated (see V. V. Beletskiy [126, 127]). Case f) may occur when $c_1 = 0$, $h_1 = -c_2$, and consequently, meridional orbits exist in this case also.

In case c) from the condition $0 < |\mu_1| < 1$ we have the inequality

$$2c_2 + c_1^2 < 0, \tag{1.8}$$

which will be made use of later on in the analysis of the variation of the elliptical coordinate λ.

Thus, the motion of artificial earth satellites along trajectories of double curvature is possible only in case c). Representing $f(\mu)$ in the form

$$f(\mu) = 2h_1 (\mu^2 - \mu_1^2)(\mu^2 - \mu_2^2), \tag{1.9}$$

we transform equation (1.1) to the following form:

$$\int \frac{d\mu}{\sqrt{(\mu^2 - \mu_1^2)(\mu^2 - \mu_2^2)}} = \sqrt{2h_1}(\tau + c_3). \tag{1.10}$$

Inverting the elliptical integral (1.10), we obtain an explicit expression for μ as a function of regularized time:

$$\mu = \mu_2 \operatorname{sn} \sigma (\tau - \tau_0), \tag{1.11}$$

where the symbol τ_0 denotes the constant of integration, and σ is defined by the formula

$$\sigma = \mu_1 \sqrt{2h_1}. \tag{1.12}$$

The modulus of the Jacobi function is equal to

$$k = \frac{\mu_2}{\mu_1}. \tag{1.13}$$

2. Study of the coordinate λ. Let us denote the polynomial in the right-hand portion of formula (1.2) by the symbol $\psi(\lambda)$, so that

$$\psi(\lambda) = -2h_1\lambda^4 - \frac{2fM}{c^3}\lambda^3 + 2(c_2 - h_1)\lambda^2 - \frac{2fM}{c^3}\lambda + (2c_2 + c_1^2), \tag{1.14}$$

and let λ_1, λ_2, λ_3, λ_4 be the roots of this polynomial.

According to the inequality (1.8) all of the coefficients of polynomial (1.14) are negative and, consequently, all of its real roots are also negative. Depending upon the relationship between the coefficients, the following cases may be distinguished:

a) $\psi(\lambda)$ has only complex roots, (1.15)
b) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 < 0$, (1.16)
c) $\lambda_1 = \lambda_2 = \lambda_3 < \lambda_4 < 0$, (1.17)
d) $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$, (1.18)
e) $\lambda_1 < \lambda_2 = \lambda_3 = \lambda_4 < 0$, (1.19)
f) $\lambda_1 = \lambda_2 < \lambda_3 = \lambda_4 < 0$, (1.20)
g) $\lambda_1 = \lambda_2 < \lambda_3 < \lambda_4 < 0$, (1.21)
h) $\lambda_1 < \lambda_2 \quad \lambda_3 < \lambda_4 < 0$, (1.22)
i) $\lambda_1 < \lambda_2 < \lambda_3 = \lambda_4 < 0$. (1.23)
j) $\lambda_1 = \lambda_2 < 0$, λ_3, λ_4 are complex, (1.24)
k) $\lambda_1 < \lambda_2 < 0$, λ_3, λ_4 are complex. (1.25)

As will be evident later on, real motion is impossible in certain of the indicated cases. In order to distinguish these cases we establish a criterion for the existence of real motion. From the tranformation formulas (6.39) /164
Chapter 3, we can obtain (see also (3.20), Chapter I)

$$\frac{x^2 + y^2}{c^2(1+\lambda^2)} + \frac{z^2}{c^2\lambda^2} = 1,$$

from which, following simple estimates, we arrive at

$$c^2(1+\lambda^2) > R^2, \tag{1.26}$$

where R is the polar radius of the earth. Substituting the value of c from Table 3, § 6, Chapter 3 in the latter inequality, we arrive at the following criterion for the existence of real motion:

$$|\lambda| > 30. \tag{1.27}$$

For convenience of analysis, we shall introduce in place of (1.14) the polynomial

$$\varphi(\lambda) = \lambda^4 + \bar{a}\lambda^3 + \bar{b}\lambda^2 + \bar{c}\lambda + \bar{d}, \tag{1.28}$$

which has the same roots as $\psi(\lambda)$. Its coefficients are

$$\bar{a} = \frac{fM}{h_1 c^3}, \quad \bar{b} = 1 - \frac{c_2}{h_1}, \quad \bar{c} = \frac{fM}{h_1 c^3}, \quad \bar{d} = -\frac{2c_2 + c_1^2}{2h_1} \tag{1.29}$$

For elliptical orbits of double curvature these coefficients are positive, and by reason of (1.8) the inequality

$$\bar{b} > \bar{a} + 1, \tag{1.30}$$

holds.

Let us analyze various cases in detail.

Case a). Real motions are absent, since here $\psi(\lambda) < 0$.

Case b). $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 < 0$. As is readily evident, the motions must take place on the ellipsoid

$$\lambda = \lambda_1.$$

However, taking into consideration the equality of the coefficients of λ^3 and λ in polynomial (1.26), with the help of Vieta's theorem we obtain

$$\lambda_1^3 = \lambda_1, \tag{1.32}$$

from which it follows that λ_1 may assume the values -1, 0, and +1. Since such values of λ do not satisfy the criterion (1.27), it follows that case b) is eliminated.

Case c). $\lambda_1 = \lambda_2 = \lambda_3 < \lambda_4 < 0$. Vieta's theorem gives /165

$$\left.\begin{aligned} 3\lambda_1 + \lambda_4 &= -\bar{a}, \\ \lambda_1^3 + 3\lambda_1^2\lambda_4 &= -\bar{a}, \end{aligned}\right\} \tag{1.33}$$

from which it follows that

$$\frac{\lambda_1}{\lambda_2} = \frac{1-3\lambda_1^2}{\lambda_1^2-3}. \tag{1.34}$$

From the condition that all roots of $\phi(\lambda)$ are negative, we obtain the inequality

$$\frac{1-3\lambda_1^2}{\lambda_1^2-3} > 0, \tag{1.35}$$

the solution of which gives

$$-\sqrt{3} < \lambda_1 < -\frac{1}{\sqrt{3}}. \tag{1.36}$$

Since the reason of variation of coordinate λ is determined by the inequality $\lambda_1 \leq \lambda \leq \lambda_4$, then λ does not exceed $\sqrt{3}$, but this condition contradicts the existence criterion (1.27). Thus this particular case is also eliminated.

Case d). We shall first demonstrate that two roots of the polynomial $\phi(\lambda)$ belong to the interval (-1,0). Compiling Shturm's system [128] for the polynomial $\phi(\lambda)$:

$$\left.\begin{aligned} \varphi_1 &= 4\lambda^3 + 3\bar{a}\lambda^2 + 2\bar{b}\lambda + \bar{a}, \\ \varphi_2 &= a_2\lambda^2 + b_2\lambda + d_2, \\ \varphi_3 &= a_3\lambda + b_3, \\ \varphi_4 &= a_4. \end{aligned}\right\} \tag{1.37}$$

where the coefficients a_i, b_i, d_i are equal to

$$\left.\begin{aligned} a_2 &= \frac{1}{16}(3\bar{a}^2 - 8\bar{b}), \quad b_2 = \frac{1}{8}\bar{a}(\bar{b}-6), \quad d_2 = \frac{1}{16}(\bar{a}^2 - 16\bar{d}), \\ a_3 &= 8a_2^{-2}(-3\bar{a}^4 + \bar{a}^2\bar{b}^2 + 14\bar{a}^2\bar{b} - 18\bar{a}^2 - 4\bar{b}^3 - 6\bar{a}^2\bar{d} + 16\bar{b}\bar{d}), \\ b_3 &= 16a_2^{-2}(\bar{a}^3\bar{b} + 3\bar{a}^3 - 9\bar{a}^3\bar{d} + 32\bar{a}\bar{b}\bar{d} - 48\bar{a}\bar{d} - 4\bar{a}\bar{b}^2), \\ a_4 &= a_2^{-3}(a_3b_2b_3 - a_3^2d_2 - a_2b_3^2), \end{aligned}\right\} \tag{1.38}$$

with the help of Shturm's theorem we conclude that all roots of the polynomial $\phi(\lambda)$ are real and different for the condition /166

$$a_2 > 0, \quad a_3 > 0, \quad a_4 > 0. \tag{1.39}$$

By converting to polar coordinates $\bar{a} = r'\cos\phi'$, $\bar{b} = r'\sin\phi'$ inequalities (1.39) with allowance for (1.38), are transformed to the following form:

$$\left.\begin{aligned} a_2 &= 3r'^2\cos^2\varphi' - 8r'\sin\varphi' > 0, \\ a_3 &= \sum_{k=0}^{8} b_k(\varphi')\,r'^k + b_9(\varphi')\,r'^{10} > 0, \end{aligned}\right\} \tag{1.40}$$

while the coefficients necessary in the subsequent analysis are equal to

$$\left.\begin{aligned} a_0(\varphi') &= \cos^2\varphi'(1 - 4\cos^2\varphi'), \\ b_0(\varphi') &= 9\cos^8\varphi'(1 - 5\cos^2\varphi'). \end{aligned}\right\} \tag{1.41}$$

If r' is sufficiently large, then with

$$a_0(\varphi') > 0, \qquad b_0(\varphi') > 0 \tag{1.42}$$

inequalities (1.40), and hence also (1.39), are fulfilled. Inequalities (1.42) also may be satisfied, provided the constants $\bar{a}$ and $\bar{b}$ are chosen from the condition

$$\arctan 2 < \phi' < \frac{\pi}{2}. \tag{1.43}$$

However, from (1.38) and (1.27) it is evident that it is always possible to chose the initial conditions in such a way that the constants of integration h and c_2 will assure fulfillment of inequality (1.43). Consequently, real forms of motion always correspond to the case in question. Moreover, by reason of the theorem concerning the upper limit of the moduli of the roots of the polynomials [128], for an appropriate choice of values of total energy and the constant of areas, the moduli of the roots λ_i may assume arbitrarily large values. It now remains only to determine whether satellite-type orbits are possible in case d).

For this purpose we estimate the interval in which the roots λ_3 and λ_4 are included. We apply the Boudin-Fourier [128] in the interval (-1, 0). Computing successfully the derivatives from $\phi(\lambda)$ to the fourth order inclusive, /167 we find that with $\lambda = 0$

$$\varphi(0) = 0,\ \varphi'(0) > 0,\quad \varphi''(0) > 0,\ \varphi'''(0) > 0,\ \varphi^{IV}(0) > 0.$$

If $\lambda = -1$, then it is necessary that $\phi(-1) > 0$, since otherwise condition (1.43) would not be satisfied. For real motion, moreover, it is necessary that $\phi'''(-1) > 0$, i.e., $\bar{a} > 4$, for in the contrary meaning of the inequality ($\phi'''(-1) < 0$) we would have $\bar{a} < 4$, $\bar{b} < 6$, $\bar{d} < 5$, which would be a contradiction of the criterion of existence of real orbits. The case in which the function $\phi(\lambda)$ and its four derivatives up to the fourth degree for $\lambda = -1$ are positive, also drops out, since here $\bar{a} < 6$, which is impossible on the basis of (1.27).

Consequently, we have one of the variants indicated in Table 4. By verifying the corresponding inequalities, we find that all three variants are possible. Thus, roots λ_3 and λ_4 are real within the interval (-1, 0).

TABLE 4

φ (−1)	+	+	+
φ' (−1)	+	−	−
φ'' (−1)	−	−	+
φ''' (−1)	+	+	+
φ^{IV} (−1)	+	+	+

The process of obtaining an equation which will constitute an explicit expression for the coordinate λ in terms of τ is carried out in the following manner. Integrating (1.2) and transforming the resulting elliptic integral (naturally making allowance for the relationships between the roots λ_i), we obtain for the spheroidal coordinate λ the following:

$$\lambda = \frac{A + B\,\mathrm{sn}^2\sigma_1(\tau - \tau_1)}{C + D\,\mathrm{sn}^2\sigma_1(\tau - \tau_1)}, \tag{1.44}$$

in which the following notations are adopted

$$\left.\begin{aligned} A &= (\lambda_3 - \lambda_1)\lambda_2, \quad B = (\lambda_1 - \lambda_2)\lambda_3, \\ C &= \lambda_3 - \lambda_1, \qquad\quad D = \lambda_1 - \lambda_2, \end{aligned}\right\} \tag{1.45}$$

$$\sigma_1 = \sqrt{0{,}5h(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_4)}. \tag{1.46}$$

The modulus of the Jacobi function is defined as follows: /168

$$k_1^2 = \frac{(\lambda_4 - \lambda_3)(\lambda_2 - \lambda_1)}{(\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)} \tag{1.47}$$

Finally, we obtain the following value for the coordinate w by means of quadrature:

$$w = c_5 + c_1 \int \frac{(\lambda^2 + \mu^2)\,d\tau}{(1 - \mu^2)(1 + \lambda^2)} \tag{1.48}$$

which can also be written as follows:

$$w = c_5 + c_1 I_1 - c_1 I_2, \tag{1.49}$$

where the following notations are adopted

$$I_1 = \int \frac{d\tau}{1 - \mu^2}, \tag{1.50}$$

$$I_2 = \int \frac{d\tau}{1 + \lambda^2}. \tag{1.51}$$

In the present instance as well as in all other cases of orbits of the elliptic class, the integral (1.50), which is an elliptic integral of the third type, is transformed in the same way. As a result of the transformations it is possible to obtain the following expression for this integral:

$$I_1 = \frac{1}{\sigma} \Pi (\phi, k, n), \tag{1.52}$$

where $\Pi (\phi, k, n)$ denotes the normal Legendre form of the elliptic integral of third type, while

$$\varphi = \operatorname{am}\sigma (\tau - \tau_0), \tag{1.53}$$

$$n = -\mu_2^2. \tag{1.54}$$

Next, substituting $x = \operatorname{sn}\sigma_1 (\tau - \tau_1)$, integral (1.51) is transformed to

$$I_2 = \frac{\tau}{1 + \lambda_3^2} + \int_0^x \frac{(a_0 x^2 + a_1)\, dx}{(b_0 x^4 + b_1 x^2 + b_2) \sqrt{(1 - x^2)(1 - k_1^2 x^2)}}, \tag{1.55}$$

in which we have the following values for the coefficients a_i, b_i:

$$\left.\begin{aligned} a_0 &= \frac{2BD(BC - AD)}{\sigma_1 (B^2 + D^2)}, \quad a_1 = \frac{B^2C^2 - A^2D^2}{\sigma_1 (B^2 + D^2)}, \\ b_0 &= B^2 + D^2, \qquad b_1 = 2(AB + CD) \\ & \qquad b_2 = A^2 + C^2. \end{aligned}\right\} \tag{1.56}$$

The integral in formula (1.55) can be expressed in terms of two elliptic integrals of third type with complex-conjoint parameters, which in turn can be transformed to elliptic intervals of third type with real parameters [129]. /169

In the calculations we can make use of (1.55), or, what is much simpler, employ rapidly converging expansions of elliptic functions and integrals in series (the rapid convergence in the case of satellite motion is guaranteed by the smallness of the moduli of the elliptic functions). These particular concepts of the coordinates have been advanced by I. Izzak [130] and Ye. P. Aksenov [131-134].

Here we have pointed out only the final expressions, and not the complete expressions for all the coefficients. Hoüel's transformation [135] (see also [129]) gives

$$I_2 = \frac{\tau}{1+\lambda_3^2} + l_1 L_1 + l_2 L_2, \tag{1.57}$$

where the functions L_1 and L_2 have the following form:

$$L_i = \int \frac{dx}{h_i x^2 + 1} - \frac{1}{g_i} F[\operatorname{am} \sigma_1(\tau - \tau_1)] - Q_i \Pi[\operatorname{am} \sigma_1(\tau - \tau_1), k, n_i] \quad (i = 1, 2), \tag{1.58}$$

the quantities l_1, l_2, g_i, n_i, and Q_i are expressed in terms of the constants determined by formulas (1.56).

Thus, we finally obtain the following expression for right ascension:

$$\omega = c_5 - \frac{c_1 \tau}{1+\lambda_3^2} + \frac{c_1}{\sigma} \Pi(\varphi, k, n) - c_1 l_1 L_1 - c_1 l_2 L_2. \tag{1.59}$$

The equation for time, which expresses the relationship between true and regularized time, has the following form:

$$t = c_6 + (\mu_1^2 + \lambda_3^2)\tau - \frac{\mu_1^2}{\sigma} E(\varphi, k) + l_1 \bar{L}_1 + l_2 \bar{L}_2, \tag{1.60}$$

where $E(\phi, k)$, as before, is an elliptic integral of the second type, while the functions L_1 and L_2 are given by formulas similar to (1.58).

In case d) the orbit is located between two confocal ellipsoids $\lambda = \lambda_1$ and $\lambda = \lambda_2$ outside the hyperboloid $\mu = \mu_2$ (Figure 22). If the $|\lambda_2| < 30$, then the orbits will be ballistic, since the inner ellipsoid will be located within the body of the earth (it is interesting to compare this with case e). Satellite-type trajectories are successively tangent to the boundaries of the ellipsoids. During various loops the points of tangency produce "relative" pericenters and apocenters. We should note also that according to § 10 Chapter I and § 6 Chapter 3, the motions will be conditional-periodic in character. If, in addition, the simple periods are incommensurable, then the trajectory will everywhere densely fill the region of possible motion indicated in Figure 22. /170

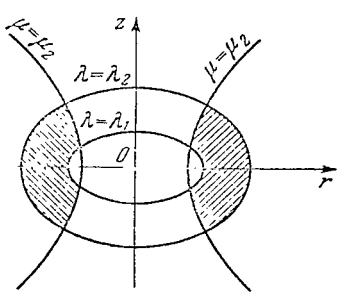

Figure 22.

Case e). Here, according to Vieta's theorem,

$$\left.\begin{aligned} \lambda_1 + 3\lambda_2 &= -\bar{a}, \\ 3\lambda_1\lambda_2 + 3\lambda_2^2 &= \bar{b}, \\ 3\lambda_1\lambda_2^2 + \lambda_2^3 &= -\bar{a}, \\ \lambda_1\lambda_2^3 &= \bar{d}. \end{aligned}\right\} \tag{1.61}$$

From the first and the third of the equations of (1.61) it follows that real motions exist, provided

$$\frac{\lambda_2(\lambda_2^2-3)}{1-3\lambda_2^2} < -30. \tag{1.62}$$

These inequalities are satisfied if

$$0{,}56 < \lambda_2 < 0{,}58, \quad \lambda_2 > 90{,}5.$$

But, since λ_2 must be negative, $-0.59 < \lambda_2 < -0.58$. For such values of λ_2 the inequality

$$\overline{b} > \overline{d} + 1,$$

is satisfied.

Thus we see that only ballistic trajectories are possible in case e). It may be pointed out that the parametric equations of the orbit have the following form: /171

$$\left.\begin{aligned} \mu &= \mu_2 \operatorname{sn}\sigma(\tau-\tau_0) \\ \lambda &= \lambda_2 + \frac{\lambda_1-\lambda_2}{1+\frac{h_1}{2}(\tau-\tau_1)^2}, \\ w &= c_0 - c_1\tau + \frac{c_1}{\sigma}\Pi[\operatorname{am}\sigma(\tau-\tau_0), k, n] - \frac{2c_1}{(\lambda_2-\lambda_1)\sqrt{2h_1}}\times \\ &\times\left[\frac{M_1}{2}\ln\frac{(u-p)^2+q^2}{(u+p)^2+q^2} + \frac{M_1p+N_1}{2}\operatorname{arctg}\frac{2qu}{p^2+q^2-u^2}\right], \end{aligned}\right\} \tag{1.63}$$

where the following notations are adopted:

$$u = \sqrt{\frac{h_1}{2}}(\tau-\tau_1), \tag{1.64}$$

$$\left.\begin{aligned} M_1 &= \frac{\lambda_2(\lambda_2-\lambda_1)}{2p(1+\lambda_2^2)} - \frac{\lambda_2^2-\lambda_1^2}{4p\sqrt{1+\lambda_1^2+\lambda_2^2+\lambda_1^2\lambda_2^2}}, \\ N_1 &= \frac{\lambda_2^2-\lambda_1^2}{\sqrt{1+\lambda_1^2+\lambda_2^2+\lambda_1^2\lambda_2^2}}, \end{aligned}\right\} \tag{1.65}$$

$$\left.\begin{aligned} p &= \sqrt{\frac{\sqrt{1+\lambda_1^2+\lambda_2^2+\lambda_1^2\lambda_2^2}-1-\lambda_1\lambda_2}{2(1+\lambda_2^2)}}, \\ q &= \sqrt{\frac{\sqrt{1+\lambda_1^2+\lambda_2^2+\lambda_1^2\lambda_2^2}+1+\lambda_1\lambda_2}{2(1+\lambda_2^2)}}. \end{aligned}\right\} \tag{1.66}$$

The equation of time is written in the following manner:

$$t = c_6 + (\mu_1^2 + \lambda_2^2)\tau + \frac{(\lambda_2 - \lambda_1)^2(\tau - \tau_1)}{2(u^2+1)} - \\ - \frac{\lambda_1 + 3\lambda_2}{\sqrt{2h_1}}\arctan u - \frac{\mu_1^2}{\sigma}\mathrm{E}[\mathrm{am}\,\sigma(\tau - \tau_0), k]. \tag{1.67}$$

Case f). For motions of this type the ellipsoid $\lambda = \lambda_3$ is located entirely within the body of the earth. Motion is possible only on the surface of the ellipsoid $\lambda = \lambda_1$. From Vieta's theorem it follows that /172

$$\left.\begin{aligned} 2(\lambda_1 + \lambda_3) &= -\bar{a}, \\ \lambda_1^2 + 4\lambda_1\lambda_3 + \lambda_3^2 &= \bar{b}, \\ 2\lambda_1\lambda_3(\lambda_1 + \lambda_3) &= -\bar{a}, \\ \lambda_1^2\lambda_3^2 &= \bar{d}. \end{aligned}\right\} \tag{1.68}$$

From the first and third equations of (1.68) we find that

$$\lambda_1\lambda_3 = 1. \tag{1.69}$$

Since $\lambda_1 \leq -30$, then $-\frac{1}{30} \leq \lambda_3 < 0$, i.e., the ellipsoid $\lambda = \lambda_3$ lies within the earth. From (1.68) and (1.69) it follows that $\bar{d} = 1$. Then, from the second equation of (1.68) we have $\bar{b} > 4$. From this it is not difficult to deduce the possibility of motion on the ellipsoid $\lambda = \lambda_1$. Transforming the quadratures enables us to arrive at the following relationships between the coordinates and regularized time:

$$\left.\begin{aligned} \mu &= \mu_2\,\mathrm{sn}\,\sigma(\tau - \tau_0), \\ \lambda &= \lambda_1, \\ \omega &= c_6 - \frac{c_1}{1+\lambda_1^2}\tau + \frac{c_1}{\sigma}\Pi[\mathrm{am}\,\sigma(\tau - \tau_0), k, n], \end{aligned}\right\} \tag{1.70}$$

while regularized time τ as a function of t is defined by the following transcendental equation:

$$t = c_6 + (\mu_1^2 + \lambda_1^2)\tau + \frac{\mu_1^2}{\sigma}\mathrm{E}[\mathrm{am}\,\sigma(\tau - \tau_0), k]. \tag{1.71}$$

Case g). Since $\lambda_1 = \lambda_2 < \lambda_3 < \lambda_4 < 0$, then, according to Vieta's theorem,

$$\left.\begin{aligned} 2\lambda_1+\lambda_3+\lambda_4 &= -\bar{a},\\ \lambda_1^2+2\lambda_1(\lambda_3+\lambda_4)+\lambda_3\lambda_4 &= \bar{b},\\ \lambda_1^2(\lambda_3+\lambda_4)+2\lambda_1\lambda_3\lambda_4 &= -\bar{a},\\ \lambda_1^2\lambda_3\lambda_4 &= d. \end{aligned}\right\} \tag{1.72}$$

From (1.72) we find that /173

$$\lambda_3+\lambda_4=\frac{2\lambda_1}{1-\lambda_1^2}(\lambda_3\lambda_4-1). \tag{1.73}$$

Since in real motion $\lambda_1 < -30$, then from (1.73) it follows that

$$\frac{2\lambda_1}{1-\lambda_1^2}>0$$

and, moreover,

$$\max_{\lambda_1<-30}\left|\frac{2\lambda_1}{1-\lambda_1^2}\right|\approx\frac{1}{15}. \tag{1.74}$$

With the help of (1.73) and (1.74), we find that

$$0<\lambda_3\lambda_4<1. \tag{1.75}$$

From this we derive the inequality

$$|\lambda_3+\lambda_4|<1. \tag{1.76}$$

This result admits of the following geometrical interpretation: the ellipsoid $\lambda = \lambda_1$ is located beyond the surface of the earth, while the ellipsoids $\lambda = \lambda_3$ and $\lambda = \lambda_4$ lie entirely within the body of the earth. For the case under consideration, the inequality (1.30) can be transformed into the following form:

$$\frac{1-\lambda_1^2}{2\lambda_1}>\frac{2\lambda_1}{1-\lambda_1^2}. \tag{1.77}$$

This inequality, however, is always satisfied provided $\lambda_1 < -30$.

This type of motion is therefore possible. The formulas for the ellipsoidal coordinates have the same structure as in the preceding case. However, the coefficients which appear in the formulas have a somewhat different form.

Case h). As was already noted, in this type of motion $\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4 < 0$. The relationship between the coefficients of the polynomial $\phi(\lambda)$ and its roots is indicated by the following equations: /174

$$\left.\begin{aligned} \lambda_1 + 2\lambda_2 + \lambda_4 &= -\bar{a}, \\ 2\lambda_1\lambda_2 + 2\lambda_2\lambda_4 + \lambda_2^2 + \lambda_1\lambda_4 &= \bar{b}, \\ \lambda_1\lambda_2^2 + 2\lambda_1\lambda_2\lambda_4 + \lambda_2^2\lambda_4 &= -\bar{a}, \\ \lambda_1\lambda_2^2\lambda_4 &= \bar{d}. \end{aligned}\right\} \tag{1.78}$$

From these equations we find that

$$\lambda_1 + \lambda_4 = \frac{2\lambda_2(\lambda_1\lambda_4 - 1)}{1 - \lambda_2^2}, \tag{1.79}$$

from which it follows, in view of condition (1.30), that

$$(\lambda_2^2 - 1)\ (\lambda_1\lambda_4 - 1) < 2\lambda_2(\lambda_1 + \lambda_4). \tag{1.80}$$

We shall show that in the type of motion under consideration only ballistic trajectories are possible. The contrary assumption means that either the inequality $\lambda_4 < -30$, or, at the very least, the inequality $\lambda_2 < -30$, would have to be satisfied. If the first of these inequalities holds, then the roots λ_1 and λ_2 would certainly have to be less than -30. But in that case $2\lambda_2(1 - \lambda_2^2) > 0$ and $\lambda_1\lambda_4 - 1 > 0$, which is impossible. Actually, if the latter inequality holds, then the right-hand member of (1.79) will be positive; but since the roots λ_1 and λ_4 are negative, the left-hand member of (1.79) is also negative. Consequently, the assumption that $\lambda_4 < -30$ is erroneous. We shall assume now that $\lambda_2 < -30$ and that $-30 < \lambda_4 < 0$. From this it follows that

$$0 < \frac{30\lambda_2}{1 - \lambda_2^2} < 1. \tag{1.81}$$

The relationships (1.79) and (1.80) will be satisfied only if $\lambda_1\lambda_4 - 1 < 0$. It follows from this that

$$30\lambda_2(1 - \lambda_1\lambda_4) < 1 - \lambda_2^2.$$

In this case, however, the inequality $|\lambda_1 + \lambda_4| < \frac{1}{15}$, which is impossible, since $\lambda_1 < -30$, $\lambda_2 < -30$. Thus, if real forms of motion are possible in the

case under consideration, they necessarily take place along ballistic trajectories. /175

We shall now demonstrate that for certain definite initial conditions, motions of this type are actually possible. From (1.79) we find that

$$\lambda_1 = \frac{\lambda_4(\lambda_2^2 - 1) - 2\lambda_2}{1 - \lambda_2^2 - 2\lambda_2\lambda_4}. \tag{1.82}$$

From this it is evident that for small negative values of λ_4, and also for negative values and values which are close to unity for λ_2, we will obtain values less than -30 for λ_1. It is easy to demonstrate that the inequality (1.80), is also satisfied.

Thus, in the case under consideration motion takes place between the surface of the earth and the ellipsoid $\lambda = \lambda_1$. The formulas for the ellipsoidal coordinates are as follows:

$$\left.\begin{aligned}
&\mu = \mu_2 \operatorname{sn}\sigma(\tau - \tau_0),\\
&\lambda = \frac{\lambda_4(\lambda_2 - \lambda_1)(1 - e^u)^2 + \lambda_1(\lambda_4 - \lambda_2)(1 + e^u)^2}{(\lambda_2 - \lambda_1)(1 - e^u)^2 + (\lambda_4 - \lambda_2)(1 + e^u)^2},\\
&\omega = c_4 + \frac{Ac_1}{a_1}\tau + \frac{c_1}{\sigma}\Pi[\operatorname{am}\sigma(\tau - \tau_0),\ k,\ n] -\\
&- \frac{c_1}{a_1\sqrt{2h_1(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)}}\left\{\frac{M_1}{2}\ln[(e^u - m_1)^2 + n_1^2] + \right.\\
&+ \frac{M_2}{2}\ln[(e^u - m_2)^2 + n_2^2] + \frac{M_1m_1 + N_1}{n_1}\operatorname{arctan}\frac{e^u - m_1}{n_1} +\\
&\left. + \frac{M_2m_2 + N_2}{n_2}\operatorname{arctan}\frac{e^u - m_2}{n_2}\right\},
\end{aligned}\right\} \tag{1.83}$$

where

$$u = \sqrt{2h_1(\lambda_4 - \lambda_2)(\lambda_2 - \lambda_1)}(\tau - \tau_1), \tag{1.84}$$

while m_1, m_2, n_1, n_2, represent the real and the imaginary portions of the equation

$$a_4x^4 + a_3x^3 + a_2x^2 + a_3x + a_4 = 0 \tag{1.85}$$

in which /176

$$\begin{aligned}
a_2 &= 6 - 4\bar{a}^2 + 6\bar{a}^4 + 6\lambda_4^2 - 4\bar{a}^2\lambda_1\lambda_4 + 6\bar{a}^4\lambda_1^2,\\
a_3 &= -4 + 4\bar{a}^2 - 4\lambda_4^2 + 4\bar{a}^4\lambda_1^2,\\
a_4 &= (1 - \bar{a}^2)^2 + (\lambda_4 + a^2\lambda_1)^2.
\end{aligned}$$

The constants A, M_1, M_2, N_1, N_2, are given by the formulas

$$\begin{aligned}
&A + M_1 + M_2 = (1 + \bar{a}^2)^2,\\
&-2(m_1 + m_2)A - 2m_2M_1 - 2m_1M_2 + N_1 + N_2 =\\
&\qquad = 4(\bar{a}^4 - 1),\\
&(m_1^2 + m_2^2 + 4m_1m_2 + n_1^2 + n_2^2)A + (m_2^2 + n_2^2)M_1 +\\
&+ (m_1^2 + n_1^2)M_2 - 2m_2N_1 - 2m_1N_2 = 6 - 4\bar{a}^2 + 6\bar{a}^4,\\
&[-2m_1(m_2^2 + n_2^2) - 2m_2(m_1^2 + n_1^2)]A +\\
&\qquad + (m_2^2 + n_2^2)N_1 + (m_1^2 + n_1^2)N_2 = 4(\bar{a}^4 - 1),\\
&(m_1^2 + n_1^2)(m_2^2 + n_2^2)A = (\bar{a}^2 + 1)^2.
\end{aligned} \tag{1.86}$$

The time equation is written:

$$\begin{aligned}
t = c_6 + (\mu_1^2 + \lambda_2^2)\tau - \frac{\mu_1^2}{\sigma}\mathrm{E}[\mathrm{am}\,\sigma(\tau - \tau_0),\ k] +\\
+ P_1 \frac{se^u - 1}{e^{2u} - 2se^u + 1} + Q_1 \arctan\frac{e^u - s}{r},
\end{aligned} \tag{1.87}$$

where

$$\begin{aligned}
&P_1 = \frac{1}{8\sqrt{2h_1}(\lambda_2 - \lambda_1)^{1/2}(\lambda_4 - \lambda_2)^{3/2}}[-16\lambda_2^2\lambda_4^2 +\\
&+ 16\lambda_1^2\lambda_2^2 + 16\lambda_2^4 + 16\lambda_1^2\lambda_4^2 - 32\lambda_1\lambda_2^3 - 32\lambda_2^3\lambda_4 +\\
&\qquad + 64\lambda_1\lambda_2^2\lambda_4 - 32\lambda_1\lambda_2\lambda_4^2 - 32\lambda_1^2\lambda_2\lambda_4],\\
&Q_1 = \frac{2(2\lambda_2 - \lambda_1 - \lambda_4)}{\sqrt{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)}}P_1 - \frac{4\lambda_2}{\sqrt{2h}},\\
&s = \frac{2\lambda_2 - \lambda_1 - \lambda_4}{\lambda_4 - \lambda_1},\quad r = \frac{2\sqrt{(\lambda_2 - \lambda_1)(\lambda_4 - \lambda_2)}}{\lambda_4 - \lambda_1}.
\end{aligned} \tag{1.88}$$

Case i). Since the roots of the polynomial $\phi(\lambda)$ satisfy the inequalities $\lambda_1 < \lambda_2 < \lambda_3 = \lambda_4 < 0$, then, according to Vieta's theorem, the following relations must hold: /177

$$\left.\begin{aligned}
\lambda_1 + \lambda_2 + 2\lambda_3 &= -\bar{a},\\
\lambda_1\lambda_2 + 2\lambda_3(\lambda_1 + \lambda_2) + \lambda_3^2 &= \bar{b},\\
2\lambda_1\lambda_2\lambda_3 + \lambda_3^2(\lambda_1 + \lambda_2) &= -\bar{a},\\
\lambda_1\lambda_2\lambda_3^2 &= \bar{d},
\end{aligned}\right\} \tag{1.89}$$

from which we find that

$$\lambda_1+\lambda_2=\frac{2\lambda_3(\lambda_1\lambda_2-1)}{1-\lambda_3^2},\quad \lambda_1+\lambda_2<\frac{(\lambda_3^2-1)(\lambda_1\lambda_2-1)}{2\lambda_3}. \tag{1.90}$$

We shall demonstrate that the ellipsoid $\lambda = \lambda_3$ lies within the earth. If we assume the contrary, then the inequality $\lambda_3 < -30$ leads to a second inequality $0 < \lambda_1\lambda_2 < 1$. But this second inequality is impossible, since $\lambda_1 < \lambda_2 < \lambda_3$, and, consequently $\lambda_1 < -30$, $\lambda_2 < -30$, which contradicts the assumption. Moreover, the relationships (1.90) show that if the multiple root λ_3 is taken over the segment [-1, 0], then these relationships are fulfilled, and at least the root λ_1, given the appropriate choice of initial conditions, will be less than -30. If both the roots are less than -30, then satellite-type motions exist. In this case, the motion will take place within the ellipsoidal layer $\lambda_1 \leq \lambda \leq \lambda_2$.

Transforming the quadratures for λ and μ, and calculating the integrals which appear in the formula for w, we have the following:

$$\left.\begin{aligned}
&\mu=\mu_2\,\mathrm{sn}\,\sigma(\tau-\tau_0),\\
&\lambda=\frac{A+B\cos u}{C+D\cos u},\\
&w=c_5-\frac{c_1\tau}{1+\lambda_3^2}+\frac{c_1}{\sigma}\Pi[\mathrm{am}\,\sigma(\tau-\tau_0),\ k,\ n]+\\
&+\frac{2c_1}{\sqrt{2h(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}}\left\{\frac{M_1}{2}\ln[(\tan u-m_1)^2+n_1^2]+\right.\\
&+\frac{M_2}{2}\ln\left[\left(\tan\frac{u}{2}-m_2\right)^2+n_2^2\right]+\frac{M_1m_1+N_1}{n_1}\arctan\frac{\tan\frac{u}{2}-m_1}{n_1}+\\
&\left.+\frac{M_2m_2+N_2}{n_2}\arctan\frac{\mathrm{tg}\,\frac{u}{2}-m_2}{n_2}\right\},
\end{aligned}\right\} \tag{1.91}$$

where /178

$$u=\sqrt{2h(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\,(\tau-\tau_1), \tag{1.92}$$

$$\left.\begin{aligned}
A&=\lambda_3(\lambda_1+\lambda_2)-2\lambda_1\lambda_2, & B&=\lambda_3(\lambda_2-\lambda_1),\\
C&=2\lambda_3-\lambda_1-\lambda_2, & D&=\lambda_2-\lambda_1.
\end{aligned}\right\} \tag{1.93}$$

The coefficients M_1, M_2, N_1, N_2 are defined by the following system of equations:

$$\left.\begin{aligned} M_1 + M_2 &= 0, \\ -2m_2M_1 - 2m_1M_2 + N_1 + N_2 &= \bar{A}, \\ M_1(m_2^2 + n_2^2) + M_2(m_1^2 + n_1^2) - 2m_2N_1 - 2m_1N_2 &= 0, \\ N_1(m_2^2 + n_2^2) + N_2(m_1^2 + n_1^2) &= \bar{B}, \end{aligned}\right\} \tag{1.94}$$

in which

$$\left.\begin{aligned} \bar{A} &= \frac{B^2C(C-2D) - A^2D(D-2B)}{B^2 + D^2}, \\ \bar{B} &= \frac{B^2C(C+2D) - A^2D(D+2B)}{B^2 + D^2}, \end{aligned}\right\} \tag{1.95}$$

$$\left.\begin{aligned} m_1 = -m_2 &= \sqrt{\frac{(\lambda_1 - \lambda_3)\left(\sqrt{1 + \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2} - 1\right)}{2(\lambda_2 - \lambda_3)(1 + \lambda_1^2)}}, \\ n_1 = -n_2 &= \sqrt{\frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)\left(\sqrt{1 + \lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2} + 1\right)}{2(\lambda_3 - \lambda_2)(1 + \lambda_1^2)(\lambda_1^2 + \lambda_2^2 + \lambda_1^2\lambda_2^2)}}. \end{aligned}\right\} \tag{1.96}$$

The equation of time is written as follows:

$$t = c_6 + (\mu_1^2 + \lambda_1^2)\tau - \frac{\mu_1^2}{\sigma}\mathrm{E}[\mathrm{am}\,\sigma(\tau - \tau_0), k] + \\ + \frac{1}{4\sqrt{2h_1}(\lambda_3 - \lambda_1)^{3/2}(\lambda_3 - \lambda_2)^{3/2}}\left\{\frac{A_2}{C + D\cos u} + B_2 \arctan\left(s_1 \tan\frac{u}{2}\right)\right\}, \tag{1.97}$$

where

$$\left.\begin{aligned} A_2 &= -\frac{(AD - BC)^2}{D}, \quad s_1 = \sqrt{\frac{C-D}{C+D}}, \\ B_2 &= \frac{2}{\sqrt{C^2 - D^2}}\left[\frac{C(A^2D^2 - B^2C^2)}{D^2} - 2(ABD - B^2C)\right]. \end{aligned}\right\} \tag{1.98}$$

Case j). It is assumed that $\lambda_1 = \lambda_2$, while the roots λ_3 and λ_4 are imaginary. /179

In this type of motion ellipsoidal trajectories exist, which lie upon the ellipsoid $\lambda = \lambda_1$. As before, we make use of the relationship between the roots and the coefficients of the polynomial $\phi(\lambda)$:

$$\left.\begin{aligned} 2\lambda_1 + 2p &= -\bar{a}, \\ \lambda_1^2 + 4\lambda_1 p + (p^2 + q^2) &= \bar{b}, \\ 2\lambda_1^2 p + 2\lambda_1 (p^2 + q^2) &= -\bar{a}, \\ \lambda_1^2 (p^2 + q^2) &= \bar{d}, \end{aligned}\right\} \tag{1.99}$$

where, instead of λ_3 and λ_4, we introduce the quantities p and q:

$$\lambda_3 = p + iq, \ \lambda_4 = p - iq.$$

From (1.99) we find

$$p = \frac{\lambda_1}{1 - \lambda_1^2}(p^2 + q^2 - 1), \tag{1.100}$$

$$(1 - \lambda_1^2)(p^2 + q^2 - 1) > -4p\lambda_1. \tag{1.101}$$

The latter inequality follows from the condition $\bar{b} > \bar{d} + 1$. This inequality can be transformed as follows:

$$\frac{(1 - \lambda_1^2)(p^2 + q^2 - 1)^2}{\lambda_1 (p^2 + q^2 - 1)} < -4p. \tag{1.102}$$

If $\lambda_1 < -30$, then the fraction $\frac{1 - \lambda_1^2}{\lambda_1}$ is positive, and therefore

$$\frac{(p^2 + q^2 - 1)^2}{p} < -4p. \tag{1.103}$$

This inequality is fulfilled for $p < 0$. But from (1.100) we have

$$p^2 + q^2 - 1 = \frac{p(1 - \lambda_1^2)}{\lambda_1}, \tag{1.104}$$

from which it is not difficult to arrive at the estimates

$$0 < p^2 + q^2 < 1, \tag{1.105}$$

taking into consideration that $p < 0$, while $\lambda_1 < -30$. From this it follows that /180

$$-1 < p < 0.$$

We can now conclude that inequality (1.101) holds in every instance. We have demonstrated that motion on the ellipsoid $\lambda = \lambda_1$ is possible. Formulas for the

ellipsoidal coordinates are of the same form as in case f).

Case k). Here $\lambda_1 < \lambda_2 < 0$, while the roots λ_3 and λ_4 are complex. For such initial conditions, both satellite-type and ballistic trajectories are possible. The coefficients and the roots of the polynomial $\phi(\lambda)$ are related as follows:

$$\left.\begin{aligned} \lambda_1 + \lambda_2 + 2p &= -\bar{a}, \\ \lambda_1\lambda_2 + 2p(\lambda_1 + \lambda_2) + (p^2 + q^2) &= \bar{b}, \\ 2p\lambda_1\lambda_2 + (\lambda_1 + \lambda_2)(p^2 + q^2) &= -\bar{a}, \\ \lambda_1\lambda_2(p^2 + q^2) &= \bar{d}, \end{aligned}\right\} \tag{1.106}$$

in which the symbols p and q denote the real and the imaginary portions of the roots λ_3 and λ_4.

From the first and the third equations of (1.106), with the help of (1.30), we find

$$\left.\begin{aligned} 2p(\lambda_1\lambda_2 - 1) + (\lambda_1 + \lambda_2)(p^2 + q^2 - 1) &= 0, \\ p[(\lambda_1\lambda_2 - 1)^2 + (\lambda_1 + \lambda_2)^2] &< 0, \end{aligned}\right\} \tag{1.107}$$

from which it follows that

$$p < 0. \tag{1.108}$$

Since for satellite-type orbits $\lambda_1 < -30$, $\lambda_2 < -30$, then $\lambda_1\lambda_2 > 1$, while $\lambda_1 + \lambda_2 < 0$. But then the first of the conditions of (1.107) is possible only if

$$0 < p^2 + q^2 < 1. \tag{1.109}$$

From (1.1) - (1.3) for the type of motion under consideration we obtain the following:

$$\left.\begin{aligned} \mu &= \mu_2 \operatorname{sn} \sigma(\tau - \tau_0), \\ \lambda &= \frac{A + B \operatorname{cn} \sigma_1(\tau - \tau_1)}{C + D \operatorname{cn} \sigma_1(\tau - \tau_1)}, \end{aligned}\right\} \tag{1.110}$$

where the following designations have been adopted: /181

$$\left.\begin{aligned} A &= -m\lambda_1 - n\lambda_2, \quad B = -m\lambda_1 + n\lambda_2, \\ C &= -m - n, \qquad D = -m + n, \\ m &= \sqrt{(p-\lambda_2)^2 + q^2}, \\ n &= \sqrt{(p-\lambda_1)^2 + q^2}, \\ \sigma_1 &= \sqrt{2h_1 mn}, \end{aligned}\right\} \tag{1.111}$$

while the modulus of the Jacobi function which appears in the formula for λ, is equal to

$$k_1 = \frac{1}{2}\sqrt{\frac{(\lambda_2-\lambda_1)^2 - (m-n)^2}{mn}}. \tag{1.112}$$

From (1.109) it is evident that the modulus k_1 for satellite-type motion is a small quantity. As follows from formulas (1.110), the motion takes place between the confocal ellipsoids $\lambda = \lambda_1$ and $\lambda = \lambda_2$ outside the hyperboloid $\mu = \mu_2$.

The longitude (right ascension) of an artificial earth satellite is defined by a formula of the type of (1.49):

$$\omega = c_5 + \frac{c_1}{\sigma}\Pi(\varphi, n, k) - c_1 I_2, \tag{1.113}$$

where $\phi = \operatorname{am}\sigma(\tau - \tau_0)$, while the integral I_2, just as in case d), is represented in the form of a combination of two elliptic integrals of the third type with complex-conjoint parameters. These can be transformed into simpler form with the help of Hoüel's transformation [135]. The equation of time has the form indicated in (1.60).

Thus, our investigations support the conclusion that if $h < 0$, the motion of the satellite will take place either within the ellipsoidal layer between the surfaces $\lambda = \lambda_1$ and $\lambda = \lambda_2$, or upon the ellipsoid. The ellipsoids which define the region of possible motion are confocal, and are of small eccentricity $(1 + \lambda_1^2)^{-\frac{1}{2}}$, $(1 + \lambda_2^2)^{-\frac{1}{2}}$. All trajectories of the elliptic class lie within a bounded portion of space. Ballistic trajectories are also possible. Such trajectories occur when the inner ellipsoid lies within the body of the earth. The motion in every instance in which the polynomial $\phi(\lambda)$ is without multiple roots, will possess two periods. The period with respect to the coordinate μ will be equal to /182

$$T = 4K(k), \tag{1.114}$$

and that with respect to coordinate λ will be equal to

$$T_1 = 4K(k_1). \tag{1.115}$$

If these periods are incommensurable, then the motion will take place along an open curve of double curvature. The trajectory will everywhere densely fill the region of possible motion. For certain initial conditions, the periods for all the coordinates will be commensurable (the case of random degeneracy), and then the motion will be periodic, but the trajectory will close following a certain number of revolutions, which, generally speaking, will be large.

Fundamental for use in the theory of motion of artificially earth satellites are the cases d) and k), since these do not require a special choice of initial conditions (the polynomial $\phi(\lambda)$ has only simple roots). Case d) relates to orbits of low inclination to the terrestrial equator. Case k) is the most important: it includes satellite orbits of arbitrary inclination which exceeds a certain small quantity.

For greater convenience in the description of motion we can introduce certain of the quantities used in celestial mechanics. For example the quantity defined by the formula

$$T_{\Omega}^{(j)} = \int_{\tau_{j-1}}^{\tau_j} (\mu^2 + \lambda^2)\, d\tau, \tag{1.116}$$

where τ_{j-1} and τ_j are solutions of the equation

$$\operatorname{sn} \sigma (\tau - \tau_0) = 0, \tag{1.117}$$

we shall refer to as the draconikic period of revolution of the satellite in the j-th orbit. By τ_{j-1} and τ_j we shall understand two successive moments in time τ, between which the satellite passes from the southern hemisphere into the northern hemisphere. These moments are defined by the following formula:

$$\tau_j = \tau_0 + \frac{4}{\sigma} jK(k). \tag{1.118}$$

For each of the types of motion we obtain an eigenexpression for the draconikic period. If the motion takes place on an ellipsoid, then the formula for the draconikic period assumes a particularly simple form: /183

$$T_{\Omega}^{(j)} = (\mu_1^2 + \lambda_1^2)(\tau_j - \tau_{j-1}) - \frac{\mu_1^2}{\sigma}\{E[\operatorname{am}\sigma(\tau_j - \tau_0)] - E[\operatorname{am}\sigma(\tau_{j-1} - \tau_0)]\}. \tag{1.119}$$

Here we introduce the concepts of the quasi-perigee and the quasi-apogee. By the term quasi-perigee we shall understand the point of tangency of the trajectory with the inner ellipsoid which defines the region of possible motion. By the term quasi-apogee we shall understand the point of tangency of the trajectory with the outer ellipsoid. We shall use the term quasi-anomalistic period in order to indicate the interval of time between two successive passages of the satellite through the quasi-perigee on the i-th revolution. This period

will be defined by the formula

$$T_m^{(i)} = \int_{\tau_{i-1}}^{\tau_i} (\mu^2 + \lambda^2)\, d\tau, \tag{1.120}$$

where τ_{i-1} and τ_i are the roots of the equation

$$\frac{A + B\,\mathrm{sn}^2\sigma_1(\tau - \tau_1)}{C + D\,\mathrm{sn}^2\sigma_1(\tau - \tau_1)} = \lambda_2 \tag{1.121}$$

for case d) and the equation

$$\frac{A + B\,\mathrm{cn}\,\sigma_1(\tau - \tau_1)}{C + D\,\mathrm{cn}\,\sigma_1(\tau - \tau_1)} = \lambda_2 \tag{1.122}$$

for case k).

As is well known, the spheroidicity of the earth produces secular perturbations in the longitude of the ascending node and in the argument of the perigee. These particular quantities can be obtained easily in the following fashion. For example, variation in the latitude of the ascending node on the j-th revolution amounts to

$$\Delta\Omega^{(j)} = w(\tau_j) - w(\tau_{j-1}). \tag{1.123}$$

§ 2. Motions of Parabolic Type

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For the family of parabolic type h = 0, the coordinate μ is determined from the following differential equation:

$$\left(\frac{d\mu}{d\tau}\right)^2 = 2c_2\mu^2 - (2c_2 + c_1^2). \tag{2.1}$$

As follows from the formulas for the transformation of coordinates (6.39) Chapter 3, $|\mu| \leq 1$. Making due allowance for this condition, we are able to conclude that real motions exist, provided the following inequality is satisfied:

$$2c_2 + c_1^2 \leqslant 0. \tag{2.2}$$

Consequently, it follows that

$$c_2 \leq 0. \tag{2.3}$$

As a result of integrating equation (2.1), we arrive at

$$\mu = \sqrt{1 + \frac{2c_1^2}{c_2}}\sin(\sqrt{-2c_2}\,\tau + c_3), \tag{2.4}$$

where c_3 denotes the constant of integration.

From this we see that the quantities c_1 and c_2 must satisfy the inequality

$$0 \leqslant 1 + \frac{c_1^2}{2c_2} \leqslant 1. \tag{2.5}$$

This condition will be met for $c_1 = 0$ and for any value of c_2. The motion then will take place on the meridional plane (the case of polar orbits). This partial case will be considered separately in one of the succeeding paragraphs. From formulas (2.4) and (2.5) it follows that the motion takes place outside the one-sheet hyperboloid of rotation:

$$\mu = \bar{\mu} = \sqrt{1 + \frac{c_1^2}{2c_2}}. \tag{2.6}$$

If, however, both the constants c_1 and c_2 are equal to zero, then, as can be demonstrated, the trajectory will be hyperbolic and will be located either on the meridional or on the equatorial plane.

We now proceed to a study of the elliptical coordinate λ, which is defined by the equation

$$\left(\frac{d\lambda}{d\tau}\right)^2 = f(\lambda), \tag{2.7}$$

where, for the class of motion under consideration, the polynomial $f(\lambda)$ assumes the form /185

$$f(\lambda) = \bar{a}\lambda^3 + \bar{b}\lambda^2 + \bar{a}\lambda + \bar{d}, \tag{2.8}$$

whose coefficients, as follows from (2.2) and (2.3) are non-positive, and are as follows:

$$\bar{a} = -\frac{2fM}{c^3}, \quad \bar{b} = 2c_2, \quad \bar{d} = 2c_2 + c_1^2. \tag{2.9}$$

Let λ_1, λ_2, and λ_3 be the roots of the polynomial $f(\lambda)$, and assume that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3|$. Since the coefficients $\bar{a}$, $\bar{b}$, $\bar{d}$, are non-positive, then the real roots of the polynomial $f(\lambda)$ cannot be positive. Several cases may arise depending upon the initial conditions:

a) one of the roots is real, and two others are complex-conjoint;

b) λ_1, λ_2 and λ_3 are real and simple;

c) $\lambda_1 = \lambda_2 < \lambda_3 < 0$,

d) $\lambda_1 < \lambda_2 = \lambda_3 < 0$,

e) $\lambda_1 = \lambda_2 = \lambda_3 < 0$.

We shall demonstrate that the motion in all these situations takes place along unbounded orbits.

Case a). If by λ_1 we designate the real root, and assume that $\lambda_2 = m + ni$, $\lambda_3 = m - ni$, then it can be established that the following relationship between the roots and the coefficients of the polynomial obtains:

$$\left.\begin{aligned} \lambda_1 + 2m &= -\frac{\overline{b}}{\overline{a}}, \\ 2\lambda_1 m + m^2 + n^2 &= 1, \\ \lambda_1 (m^2 + n^2) &= -\frac{\overline{d}}{\overline{a}}. \end{aligned}\right\} \tag{2.10}$$

It is not difficult to demonstrate that motion is possible in case a) and that it takes place outside the ellipsoid

$$\lambda = \lambda_1. \tag{2.11}$$

Transforming the elliptic integral obtained from (2.7),

$$\int \frac{d\lambda}{\sqrt{f(\lambda)}} = \tau + c_4, \tag{2.12}$$

then we will have the following value for the coordinate λ /186

$$\lambda = \lambda_1 - \frac{[1 + \operatorname{cn}(\sigma_2 \tau + c_4)]^2}{\operatorname{sn}^2(\sigma_2 \tau + c_4)} p, \tag{2.13}$$

where c_4 is the constant of integration

$$p = \sqrt{(\lambda_1 - m)^2 + n^2}, \tag{2.14}$$

$$\sigma_2 = \sqrt{-\overline{a}p}, \tag{2.15}$$

while the modulus of the elliptic Jacobi functions is obtained from the following formula:

$$k = \sqrt{\frac{p + \lambda_1 - m}{2p}}. \tag{2.16}$$

As follows from (2.13), the coordinate λ increases without limit, that is the motion takes place in an unbounded portion of space. Below we shall demonstrate that this increase takes place along with an unlimited increase in the quantity t.

Case b). The region of possible motion is defined by the inequalities

$$\lambda \leq \lambda_1, \tag{2.17}$$

$$\lambda_2 \leq \lambda \leq \lambda_3. \tag{2.18}$$

The real forms of motion for which the conditions of (2.18) are satisfied are impossible, since neither λ_2 nor λ_3 satisfies the criterion for the existence of real motion (1.27). This follows directly from Vieta's theorem, according to which

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 1. \tag{2.19}$$

Let us assume that λ_1 (and hence also λ_2) does not exceed -30. But if this is true, then all of the terms in the left-hand portion of equation (2.19) will be positive, while their sum will exceed unity, and this is impossible. Thus, in order that equation (2.19) should be satisfied it is necessary that the roots λ_2 and λ_3 should lie within the segment [-1.0], i.e., real motion is impossible.

It remains to investigate forms of motion which satisfy the condition (2.17). Here the motion takes place outside the ellipsoid $\lambda = \lambda_1$, while for $|\lambda_1| < 30$ the region of real motion is limited by the surface of the earth. Transforming the elliptic integral (2.12) we arrive at /187

$$\lambda = \lambda_1 - \lambda_2 + \lambda_2 \operatorname{sn}^2(\sigma_3\tau + c_4), \tag{2.20}$$

where

$$\sigma_3 = \frac{1}{2}\sqrt{a(\lambda_1 - \lambda_2)}, \tag{2.21}$$

while the modulus of the Jacobi functions is determined from the formula

$$k = \sqrt{\frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1}}. \tag{2.22}$$

As follows from (2.20), the variable λ may increase indefinitely.

Case c). In this case both of the ellipsoids $\lambda = \lambda_1$ and $\lambda = \lambda_3$ lie beyond

the surface of the earth, since with $|\lambda_1| \geq 30$ the relationship (2.19) cannot apply. Thus, the motion of a point takes place outside the surface of the earth, and also outside the hyperboloid. The coordinate λ as a function of τ is defined by the following formula:

$$\lambda = \frac{\lambda_1 [1 + e^{2\sigma_4(\tau+c_4)}]^2 - 4\lambda_3 e^{2\sigma_4(\tau+c_4)}}{[1 - e^{2\sigma_4(\tau+c_4)}]^2}, \tag{2.23}$$

in which

$$\sigma_4 = \frac{1}{2}\sqrt{a(\lambda_1 - \lambda_3)}. \tag{2.24}$$

Case d). As in the preceding case, it is easy to demonstrate that the ellipsoid $\lambda = \lambda_3$ lies within the body of the earth. The restriction $\lambda < \lambda_1$ corresponds to real motion, and then from (2.12) we find that

$$\lambda = \frac{\lambda_3 [1 + e^{2\sigma_4(\tau+c_4)}]^2 - 4\lambda_1 e^{2\sigma_4(\tau+c_4)}}{[1 - e^{2\sigma_4(\tau+c_4)}]^2}. \tag{2.25}$$

In cases c) and d), as is evident from formulas (2.23) and (2.25), the coordinate λ increases indefinitely along with τ.

Case e). Here $\lambda_1 = \lambda_2 = \lambda_3$ and the ellipsoid $\lambda = \lambda_1$ also is located within the body of the earth. In the case of real motion, transformation of the elliptic quadrature (2.12) yields

$$\lambda = \lambda_1 + \frac{4}{a(\tau + c_1)}. \tag{2.26}$$

Without going into the details of the calculation of the angular coordinate w, we can make a few general remarks regarding the character of its variation. /188
This coordinate is calculated from the following formula (see (1.3)):

$$w = c_5 + c_1 \int \frac{d\tau}{1 - \mu^2} - c_1 \int \frac{d\tau}{1 + \lambda^2}. \tag{2.27}$$

We shall adopt the following designations:

$$I_1 = \int \frac{d\tau}{1 - \mu^2}, \tag{2.28}$$

$$I_2 = \int \frac{d\tau}{1 + \lambda^2}. \tag{2.29}$$

The integral I_1 has identical forms for all types of motion:

$$I_1 = \frac{1}{c_1}\arctan\frac{\sqrt{-2c_2c_1^2}\tan(\sqrt{-2c_2}\tau + c_3)}{2c_2}, \tag{2.30}$$

whereas the other integral is different for each of the cases under consideration.

In cases a) and b) this integral is expressed in terms of elliptic integrals of the first and second types. Actually, since the quantity $(1 + \lambda^2)^{-1}$ can be expressed in the form

$$\frac{1}{1+\lambda^2} = \frac{\sum_{s=0}^{2} a_{2s}\,\mathrm{cn}^{2s}(\sigma_{2,3}\tau + c_4)}{\sum_{s=0}^{4} b_s\,\mathrm{cn}^{s}(\sigma_{2,3}\tau + c_4)}, \tag{2.31}$$

we obtain the following expression for the integral I_2:

$$I_2 = \int \frac{\sum_{s=0}^{2} a_{2s}\,\mathrm{cn}^{2s}(\sigma_{2,3}\tau + c_4)}{\sum_{s=0}^{4} b_s\,\mathrm{cn}^{s}(\sigma_{2,3}\tau + c_4)}\,d\tau, \tag{2.32}$$

in which the coefficients a_{2s} and b_s depend upon the roots of the polynomial $f(\lambda)$. Making a substitution with the variable

$$v = \mathrm{cn}\,(\sigma_{2,3}\,\tau + c_4), \tag{2.33}$$

we transform the integral of (2.32) to the following form /189

$$I_2 = -\frac{1}{\sigma_{2,3}}\int \frac{\sum_{s=0}^{2} a_{2s}v^{2s}\,dv}{\sum_{s=0}^{4} b_s v^s \sqrt{(1-v^2)(1-k^2+k^2v^2)}}. \tag{2.34}$$

The integral (2.34) can be expressed in terms of elliptic integrals of the third type [135]. If it happens that the elliptic integrals of third type have imaginary parameters, then we should make use of the formulas for transformation to elliptical integrals of the third type with real parameters.

In cases c, d and e) the integral I_2 is expressed in terms of elementary functions. In case d), for example, we obtain

$$I_2 = \frac{\tau}{1+\lambda_1^2} - \frac{4\lambda_1}{a(1+\lambda_1^2)^2}\ln[(1+\lambda_1^2)a^2(\tau+c_4)^2 + 8\lambda_1 a(\tau+c_4) + 16] + \frac{8(\lambda_1^2-1)}{(1+\lambda_1^2)^2}\operatorname{arctg}\frac{(1+\lambda_1^2)a\tau + a(1+\lambda_1^2)c_4 + 4\lambda_1}{8}. \tag{2.35}$$

Analogous remarks are justified for the equation of time,

$$t = \int (\mu^2 + \lambda^2)\, d\tau. \tag{2.36}$$

Here the integral of the function $\mu^2(\tau)$ is expressed in terms of elementary functions, and has the same form for all types of motion; whereas integration of the function $\lambda^2(\tau)$ is performed differently in every case. In cases a) and b) integrals of the third type appear in the relationship between t and τ, while in the remaining cases this relationship will contain only elementary functions. We should note also that for all types of motion the relationship between t and τ is expressed in finite form.

For example, for the simplest case d) the relationship $t(\tau)$ has the following form:

$$t = \left(\frac{\bar{\mu}^2}{2} + \lambda_1^2\right)\tau - \frac{\bar{\mu}^2}{4\sqrt{-2c_2}} \sin\left(2\sqrt{-2c_2}\tau + c_3\right) - \frac{16}{\bar{a}^2(\tau + c_4)} + \frac{8\lambda_1}{\bar{a}} \ln(\tau + c_4). \tag{2.37}$$

Additional information on orbits of the parabolic class, including polar and equatorial orbits, may be found in the work by Ye. P. Aksenov, Ye. A. Grebenikov and V. G. Demin [91]. /190

§ 3. Motions of Hyperbolic Type

The class of hyperbolic motions comprises those trajectories for which $h > 0$. As is apparent from formula (1.5), if $h > 0$ the discriminant $\Delta = (h_1 + c_2)^2 + 2h_1c_1^2$ must be non-negative, since otherwise the coordinates x, y, z would be imaginary.

Analysis of the possible forms of motion when $h > 0$ is quite cumbersome. It is sufficient to point out that, depending upon the values of the constants of integration c_1, c_2, h in the normal analysis of the quadrature (1.1) which defines the coordinate μ, as many as 22 cases may arise. It is necessary to determine for which initial conditions the roots μ_i of the polynomial (1.9) will be real or complex. The value of the modulus of the root is also an important factor. The character of the motion will depend upon whether the absolute values of the root are larger or smaller than unity. In analyzing the quadrature (1.2) it is necessary to examine 34 mathematically possible cases. Many of these cases are of no interest in the theory of the motion of artificial earth satellites for one reason or another, and can therefore be disregarded. As in the preceding paragraphs, in choosing types of motion we should take into consideration the reality criterion for orbits (1.27). The study of the hyperbolic class of motions has no specific pecularities, and it can be undertaken with the help of the theorems of higher algebra, just as was done in §§ 1 and 2 previously. Research along this line was undertaken in [125], the

basic results of which are given below[1]. We deliberately omit equatorial and meridional orbits from consideration in this section; the former can be studied in a more exact formulation of the problem without having recourse to the generalized problem of two immobile centers, and the latter will be considered in a following paragraph. /191

As shown by analysis, all trajectories in this class can be reduced to four different types, which we designate below with Roman numerals.

Type I. All trajectories are included between two single-sheet hyperboloids of rotation $\mu = \mu_1$ and $\mu = \mu_2$ (Figure 23).

Type II. To this type belong trajectories in which the motion takes place outside the hyperboloid $\mu = \mu_1$ (Figure 24).

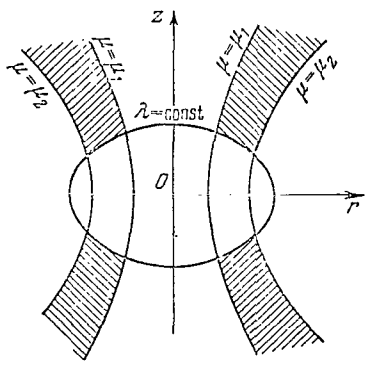

Figure 23.

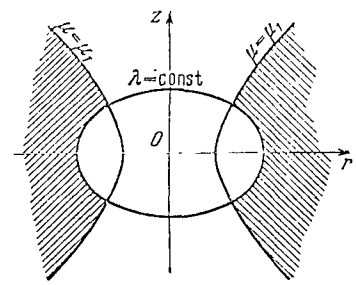

Figure 24.

Type III. Orbits of this type are stationary and lie upon the hyperboloid $\mu = \mu_1$ (Figure 25). This type of orbit may be called hyperboloidal.

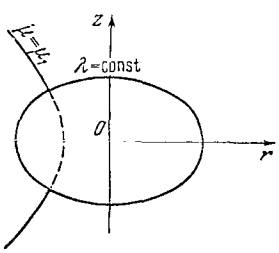

Figure 25.

Type IV. Motion takes place throughout extra-terrestrial space. There are also some forms of motion for which $\mu = \mu_1 < 1$.

We first consider the quadrature (1.1). Here we can distinguish the following cases:

Case A). μ_1 is real and greater than unity in absolute value, while μ_2 is a complex root. The polynomial $f(\mu)$ is transformed to the following form: /192

[1]The work [125] contains certain errors. Those noticed by the present author have been eliminated from §3. A number of the relationships and formulas are represented in simpler form.

$$f(\mu) = -2h_1(\mu_1^2 - \mu^2)\ (\mu^2 - 2\mu \operatorname{Re}\mu_2 + |\mu_2|^2). \tag{3.1}$$

By transforming the quadrature (1.1) we arrive at the following equation:

$$\mu = \mu_1 \operatorname{cn}[\sigma(\tau - \tau_0),\ k], \tag{3.2}$$

in which

$$\left.\begin{aligned} \sigma &= \sqrt{-2h_1(\mu_1^2 + \operatorname{Re}^2\mu_2)}, \\ k &= \frac{\mu_1}{\sqrt{\mu_1^2 + \operatorname{Re}^2\mu_2}}. \end{aligned}\right\} \tag{3.3}$$

Case B). $\mu_1 = \pm 1$, while μ_2 is a complex root. Since $\mu_1 = \pm 1$, the motion will take place in a polar orbit.

Case C). $\mu_1 = 0$, and μ_2 is a complex root. Obviously, $c_2 < -\frac{c_1^2}{2} < +h_1$: i.e., the orbits will be in the equatorial plane.

Case D). $\mu_1 = \pm 1$, $-1 \leq \mu_2 \leq 1$. This occurs when $c_1 = 0$ -- in other words the orbits will be planar and will lie in planes which are perpendicular to the equator.

Case E). $\mu_1 = \pm 1$, $\mu_2 = 0$. This occurs when $c_1 = c_2 = 0$. Transformation of the quadrature (1.1) is very simple, yielding the following:

$$\mu = \frac{1}{\operatorname{ch} u}, \tag{3.4}$$

where

$$u = \sqrt{-2h_1}(\tau - \tau_0). \tag{3.5}$$

Case F). $-1 < \mu_1 < 1$, $-1 < \mu_2 < 1$. From formula (1.5) it follows that the following conditions must be met:

$$|c_2| < -h_1,\ 2c_2 + c_1^2 > 0.$$

This means that the polynomial $f(\mu)$ can be represented in the form

$$f(\mu) = -2h_1(\mu_1^2 - \mu^2)(\mu^2 - \mu_2^2),$$

and from (1.5) we find that /193

$$\mu = \frac{\mu_1\mu_2}{\sqrt{\mu_1^2 - (\mu_1^2 - \mu_2^2)\,\mathrm{sn}^2\,\sigma_1(\tau - \tau_0)}}, \tag{3.6}$$

where

$$\sigma_1 = \mu_1\sqrt{-2h_1}, \quad k = \frac{\sqrt{\mu_1^2 - \mu_2^2}}{\mu_1^2}. \tag{3.7}$$

Case G). $-1 < \mu_1 < +1$, $\mu_2 = 0$. The roots μ_i will satisfy these conditions provided that $c_2 = -c_1^2/2 > h_1$. Since $f(\mu) = -2h_1\mu^2(\mu_1^2 - \mu^2)$, integration of (1.1) will yield

$$\mu = \frac{\mu_1}{\mathrm{ch}\,\sigma_1(\tau - \tau_0)}, \tag{3.8}$$

in which

$$\sigma_1 = \sqrt{-2h_1}. \tag{3.9}$$

Case H). The root μ_1 is real, the root μ_2 is pure imaginary, and $\mu_2 = i\mu_1$. The constants of integration must satisfy the conditions $c_2 = h_1 < -c_1^2/2$. From formula (1.1) we obtain

$$\mu = \mu_1\,\mathrm{cn}\,\sigma_2(\tau - \tau_0), \tag{3.10}$$

where

$$\sigma_2 = 2\mu_1\sqrt{-2h_1}, \quad k = \frac{1}{\sqrt{2}}. \tag{3.11}$$

Case I). If $\Delta = 0$, while $\mu_1 = \mu_2 = 0$, then the orbits will be equatorial.

Case J). If $\mu_1 = \pm 1$ or $\mu_1 = 0$, the orbits will be polar or equatorial, respectively.

Case K). $-1 < \mu_1 < 1$. If $|c_2| < -h_1$, and $2c_2 + c_1^2 > 0$, then motion along the hyperboloid $\mu = \mu_1$ is possible.

For the sake of simplicity in our study of the quadrature (1.2) we take into consideration the results of the analysis of quadrature (1.1). As has already been pointed out, all of the forms of motion can be divided into four

types. These forms of motion are characterized by the following conditions, /194
imposed upon the values of the arbitrary constants c_1, c_2 and h_1.

Type I. $2c_2 + c_1^2 > 0$, $|c_2| < -h_1$.
Type II. $2c_2 + c_1^2 < 0$.
Type III. $|c_2| < h_1$, $h_1(2c_2 + 2c_1^2 + 1) + c_2^2 = 0$.
Type IV.

$$c_2 = -\frac{c_1^2}{2} > h_1.$$

Depending upon the type of motion, there will be various interpretations of coordinate λ, which depends upon the magnitude of the roots of the polynomial $\phi(\lambda)$ (see formula (1.28)).

Type I. In this type of motion the coordinate μ is determined by formula (3.6), and the motion takes place between the hyperboloids $\mu = \mu_1$ and $\mu = \mu_2$.

Case a). If λ_1, λ_2, λ_3, λ_4 are complex, then, substituting

$$\lambda_{1,2} = m \pm in, \qquad \lambda_{3,4} = p \pm iq,$$

from (1.2) we find that

$$F\left[\alpha + \arctan\frac{\lambda - m}{n},\ k\right] = \sigma_1(\tau - \tau_1), \tag{3.12}$$

where the following designations are used:

$$\left.\begin{aligned} \sigma_1 = \frac{l - l_1}{2}\sqrt{-2h_1},\ k = \frac{2\sqrt{ll_1}}{l + l_1},\ \tan\alpha = \sqrt{\frac{4n^2 - (l - l_1)^2}{(l + l_1)^2 - 4n^2}}, \\ l^2 = (m - p)^2 + (n + q)^2,\ l_1^2 = (m - p)^2 + (n - q)^2. \end{aligned}\right\} \tag{3.13}$$

The longitude w and the equation of motion assume the following form (see formulas (6.64) and (6.65), Chapter III):

$$w = c_5 + c_1 I_{\mu w} - c_1 I_{\lambda w}, \tag{3.14}$$

$$t = c_6 - I_{\mu t} - I_{\lambda t}, \tag{3.15}$$

where

$$\left.\begin{aligned} I_{\mu w} = \int \frac{d\tau}{1 - \mu^2},\quad I_{\lambda t} = \int \lambda^2\, d\tau, \\ I_{\mu t} = \int \mu^2\, d\tau,\quad I_{\lambda w} = \int \frac{d\tau}{1 + \lambda^2}. \end{aligned}\right\} \tag{3.16}$$

The integrals of equations (3.16) can be expressed in terms of elliptic Jacobi /195
functions in finite form. By way of example we shall consider the values of the integrals $I_{\mu w}$ and $I_{\mu t}$:

$$\left.\begin{aligned} I_{\mu w} &= \tau + \frac{\mu_2^2}{1-\mu_2^2}\,\Pi\,[\operatorname{am}\sigma_1(\tau-\tau_0),\ n_1,\ k], \\ I_{\mu t} &= \frac{\mu_2^2}{\sigma_1}\,\Pi\,[\operatorname{am}\sigma_1(\tau-\tau_0),\ n_0,\ k], \end{aligned}\right\} \tag{3.17}$$

where

$$n_0 = \left(\frac{\mu_2}{\mu_1}\right)^2 - 1, \quad n_1 = \frac{n_0}{1-\mu_2^2}. \tag{3.18}$$

The expression for the other two integrals is more complex.

Case b). If $\lambda_1 = \lambda_2 > 0$, and λ_3, λ_4 are complex, then the solution of equation (1.2) is represented in the following form:

$$\lambda = \frac{n\lambda_1 \operatorname{sh}\sigma_1(\tau-\tau_1) + m\lambda_1 - (m^2+n^2)}{n\operatorname{sh}\sigma_1(\tau-\tau_1) + \lambda_1 - m}, \tag{3.19}$$

where

$$\sigma_1 = \sqrt{-2h_1[(\lambda_1-m)^2+n^2]},$$

while m and n as previously are the real and imaginary portions of the roots λ_3 and λ_4, respectively. Integrals (3.16) are expressed in terms of elementary functions.

Case c). $\lambda_1 > \lambda_2 > 0$, λ_3, λ_4 are complex. The motion takes place outside the ellipsoid $\lambda = \lambda_1$, while from (1.2) we obtain

$$\lambda = \frac{g_1 \operatorname{cn}\sigma_1(\tau-\tau_1) + g_2}{d_1 \operatorname{cn}\sigma(\tau-\tau_1) + d_2}, \tag{3.20}$$

in which the following designations are employed:

$$\left.\begin{aligned} g_1 &= q\lambda_1 + p\lambda_2, & g_2 &= q\lambda_1 - p\lambda_2, \\ d_1 &= q-p, & d_2 &= q+p, \\ p &= \sqrt{(m-\lambda_1)^2+n^2}, & q &= \sqrt{(m-\lambda_2)^2+n^2}, \\ \sigma_1 &= \sqrt{-2h_1pq}, & k_1^2 &= \frac{(p+q)^2+(\lambda_1-\lambda_2)^2}{4pq}. \end{aligned}\right\} \tag{3.21}$$

Type II. For all sub-types of motion, the expression for the coordinate μ is the same, being determined by formula (3.10). /196

Case a). If $\lambda_1 > 0$, $\lambda_2 < 0$, while λ_3, λ_4 are complex, then from equation (1.2) we obtain

$$\lambda = \frac{g_1 \operatorname{cn} \sigma_1 (\tau - \tau_1) + g_2}{d_1 \operatorname{cn} \sigma_1 (\tau - \tau_1) + d_2}, \tag{3.22}$$

where the following designations are used:

$$\left.\begin{aligned} g_1 &= q\lambda_2 + p\lambda_1, & g_2 &= q\lambda_2 - p\lambda_1, \\ d_1 &= p + q, & d_2 &= q - p, \\ p &= \sqrt{(m-\lambda_1)^2 + n^2}, & q &= \sqrt{(m-\lambda_2)^2 + n^2}, \\ \sigma_1 &= \sqrt{-2h_1 pq}, & k_1^2 &= \frac{(p+q)^2 + (\lambda_1 - \lambda_2)}{4pq}, \end{aligned}\right\} \tag{3.23}$$

while the quantities m and n are defined as previously.

The integrals (3.16) are expressed in terms of elliptic integrals of the first, second and third types. For example, for $I_{\mu w}$ and $I_{\mu t}$ we obtain the following:

$$I_{\mu t} = \frac{\mu_1^2}{\sigma} \{F[\operatorname{am} \sigma(\tau - \tau_0)] - D[\operatorname{am} \sigma(\tau - \tau_0)]\}, \tag{3.24}$$

$$I_{\mu w} = \frac{1}{\sigma(1-\mu_1^2)} \Pi\left[\operatorname{am} \sigma(\tau - \tau_0), \frac{\mu_1^2}{1-\mu_1^2}, k\right]. \tag{3.25}$$

Case b). Here $\lambda_1 > 0$, while $\lambda_2 = \lambda_3 = \lambda_4 > 0$. Transformation of the quadrature for the coordinate λ leads to the following equation:

$$\lambda = \frac{\lambda_1 - \lambda_2 \sigma_1^2 (\tau - \tau_1)^2}{1 - \sigma_1^2 (\tau - \tau_1)^2}. \tag{3.26}$$

The formula for the longitude w assumes the following form

$$w = c_5 + \frac{c_1}{\sigma(1-\mu_1^2)} \Pi\left[\operatorname{am} \sigma(\tau - \tau_0), \frac{\mu_1^2}{1-\mu_1^2}, k\right] - c_1 I_{\lambda w}, \tag{3.27}$$

while the equation of time is written as follows: /197

$$t = c_6 + \frac{\mu_1^2}{\sigma} \{D[\operatorname{am} \sigma(\tau - \tau_0), k] - F[\operatorname{am} \sigma(\tau - \tau_0), k]\} - I_{\lambda t}. \tag{3.28}$$

The integrals (3.16) may also be expressed in finite form. For example, for $I_{\lambda t}$ we have

$$I_{\lambda t} = \lambda_2 \tau + \frac{(\lambda_1 - \lambda_2)^2 (\tau - \tau_1)^2}{1 - \sigma_1^2 (\tau - \tau_1)^2} + \frac{\lambda_1^2 + 2\lambda_1\lambda_2 - 3\lambda_2^2}{4\sigma_1} \ln \frac{1 + \sigma_1 (\tau - \tau_1)}{1 - \sigma_1 (\tau - \tau_1)}. \tag{3.29}$$

Case c). $\lambda_1 > 0$, $\lambda_2 < \lambda_3 < \lambda_4 < 0$. With these values of the roots λ_i, we obtain from (1.2)

$$\lambda = \frac{\lambda_2(\lambda_1 - \lambda_3) - \lambda_3(\lambda_1 - \lambda_2)\operatorname{sn}^2\sigma_1(\tau - \tau_1)}{\lambda_1 - \lambda_3 - (\lambda_1 - \lambda_2)\operatorname{sn}^2\sigma_1(\tau - \tau_1)}, \tag{3.30}$$

where

$$k_1^2 = \frac{(\lambda_4 - \lambda_3)(\lambda_2 - \lambda_1)}{(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)}.$$

Case d). $\lambda_1 > 0$, $0 > \lambda_2 = \lambda_3 > \lambda_4$. Here the coordinate λ is expressed in terms of elementary functions:

$$\lambda = \lambda_2 - \frac{2(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{(\lambda_4 + \lambda_1 - 2\lambda_2) - (\lambda_1 - \lambda_4)\sin\sigma_1(\tau - \tau_1)}, \tag{3.31}$$

in which

$$\sigma_1 = \sqrt{-2h_1(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_4)}. \tag{3.32}$$

Case e). $\lambda_1 > 0$, $0 > \lambda_4 > \lambda_3 = \lambda_2$. For the coordinate λ we have the following formula:

$$\lambda = \lambda_2 + \frac{4(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)\exp[-\sigma_1(\tau - \tau_1)]}{[\exp(-\sigma_1(\tau - \tau_1)) + \lambda_1 + \lambda_4 - 2\lambda_3]^2 - 4(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}, \tag{3.33}$$

where

$$\sigma_1 = \sqrt{-2h_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}.$$

Case f). $\lambda_1 < 0$, $\lambda_2 = \lambda_3 = \lambda_4 > 0$. Here for λ we have the following expression:

$$\lambda = \lambda_2 - \frac{2(\lambda_2 - \lambda_1)}{4(\lambda_2 - \lambda_1)^2(\tau - \tau_1)^2 + 4}. \tag{3.34}$$

The other formulas which serve to describe this case are also fairly simple, and can be obtained rather easily. /198

Type III. As was already pointed out, this type embraces all motions which take place on the hyperboloid. Parametric equations of the orbit and the equation of time in each of the possible sub-types can be obtained from the

corresponding formulas of the previous cases by means of limiting transition. For example, in the case Ib) ($\lambda_1 = \lambda_2 > 0$, λ_3, λ_4 are complex) we have the following:

$$\mu = \mu_1, \tag{3.35}$$

$$\lambda = \frac{\lambda_2 n \operatorname{sh} \sigma_1 (\tau - \tau_1) + \lambda_1 m - (m^2 + n^2)}{n \operatorname{sh} \sigma_1 (\tau - \tau_1) + \lambda_1 - m}, \tag{3.36}$$

$$w = c_5 + \frac{c_1 \tau}{1 - \mu_1^2} - c_1 I_{\lambda w}, \tag{3.37}$$

$$t = c_6 - \mu_1^2 \tau - I_{\lambda t}. \tag{3.38}$$

If, however, $\lambda_1 = \lambda_2 < 1$, the trajectory will intersect the surface of the earth.

Type IV. Here the polynomial $\phi(\lambda)$ has a single zero root. Since $h < c_2 < 0$, the other roots must all be positive. If, for example, $\lambda_1 > 0$, $\lambda_4 = 0$, while λ_2, λ_3 are complex-conjoint, then for λ we can obtain the following formula:

$$\lambda = \frac{q\lambda_1 [1 + \operatorname{cn} \sqrt{-2h_1 pq}\,(\tau - \tau_1)]}{q - p + (q + p) \operatorname{cn} \sqrt{-2h_1 pq}\,(\tau - \tau_1)}, \tag{3.39}$$

where m, n, p, q are to be interpreted as in the preceding cases. The coordinate μ is determined by formula (3.8).

§ 4. Polar Orbits

Let us consider the polar orbits of a satellite -- i.e., those orbits which lie entirely within meridional planes[1]. From formula (1.3) § 1, it follows that for all polar orbits $c_1 = 0$. Without loss of generality we can assume /199 that the plane of a polar orbit is the coordinate plane y = 0. In accordance with (6.39) Chapter III, the constant c_5 should be set equal to $\pi/2$. Then, on the basis of (6.39) and (6.61) Chapter III, the rectangular coordinates of the satellite will be related to the ellipsoidal coordinates as follows:

$$\left.\begin{aligned} x &= c\sqrt{(1 + \lambda^2)(1 - \mu^2)}, \\ y &= 0, \\ z &= -c\lambda\mu, \end{aligned}\right\} \tag{4.1}$$

while λ and μ must be found directly by transformation of quadratures:

[1]The results given here were obtained by Ye. A. Grebenikov, Ye. P. Aksenov and the present author in [90].

$$\int \frac{d\mu}{\sqrt{2h_1\mu^4 + 2(c_2 - h_1)\mu^2 - 2c_2}} = \tau + c_3, \tag{4.2}$$

$$\int \frac{d\lambda}{\sqrt{-2h_1\lambda^4 + 2(c_2 - h_1)\lambda^2 - 2\frac{fM}{c^3}\lambda^3 - 2\frac{fM}{c^3}\lambda + 2c_2}} = \tau + c_4. \tag{4.3}$$

The latter formulas yield the following:

$$\left(\frac{d\mu}{d\tau}\right)^2 = 2h_1(\mu^2 - 1)(\mu^2 - \delta^2), \tag{4.4}$$

$$\left(\frac{d\lambda}{d\tau}\right)^2 = -2h_1(\lambda^2 - 1)\varphi(\lambda), \tag{4.5}$$

where

$$\varphi(\lambda) = \lambda^2 + \frac{fM}{h_1c^2}\lambda - \frac{c_2}{h_1}, \tag{4.6}$$

$$\delta^2 = -\frac{c_2}{h_1}. \tag{4.7}$$

As in the preceding paragraphs, we study the possible forms of satellite motion on the basis of the requirement that the right-hand portions of equations (4.4) and (4.5) must be non-negative.

From equation (4.5) it follows that we must distinguish three cases, as defined by initial conditions.

1. The roots of the polynomial $\phi(\lambda)$ are real and different.

2. The roots of the polynomial $\phi(\lambda)$ are equal.

3. The roots of the polynomial $\phi(\lambda)$ are complex-conjoint.

Designating these roots with the symbols α and β, we can write /200

$$\alpha = -\frac{fM}{2h_1c^3} + \sqrt{\left(\frac{fM}{2h_1c^3}\right)^2 + \frac{c_2}{h_1}}, \tag{4.8}$$

$$\beta = -\frac{fM}{2h_1c^3} - \sqrt{\left(\frac{fM}{2h_1c^3}\right)^2 + \frac{c_2}{h_1}}. \tag{4.9}$$

Case 1. From formula (4.5) it is evident that if $h_1 > 0$, motion is possible within the region $\beta \leq \lambda \leq \alpha$, while if $h_1 < 0$, there are two possible regions of motion, namely $\lambda > \alpha$ and $\lambda < \beta$. If $h_1 > 0$, then the motion takes place within the bounded region of space between the ellipsoids $\lambda = \alpha$ and $\lambda = \beta$. If, however, $h_1 < 0$, then the motion takes place within an unbounded region.

Case 2. The roots will be equal ($\alpha = \beta$) only upon the condition that

$$\frac{c_2}{h_1} = -\left(\frac{fM}{2h_1c^3}\right)^2, \tag{4.10}$$

in which case

$$\alpha = \beta = -\delta = -\frac{fM}{2h_1c^3}. \tag{4.11}$$

Since for satellite orbits the $|\beta| > 1$, and therefore $\delta > 1$. Consequently, if $h_1 < 0$, the right-hand member of equation (4.4) is non-negative only if $|\mu| > 1$. In real situations, however, $|\mu| \leqslant 1$. Therefore, if $h_1 < 0$, motion is impossible. If $h_1 > 0$, then the right-hand member of equation (4.4) is non-negative for all values of μ between -1 and +1. From equation (4.5) it follows that in the case of multiple roots λ remains constant during the process of motion, while it is equal to the quantity $\lambda = -fM/2h_1c^3$.

Case 3. If the roots of the polynomial $\phi(\lambda)$ are complex, then the right-hand member of equation (4.5) remains non-negative for all values of λ, provided $h_1 < 0$; conversely, for $h_1 > 0$, it assumes only negative values. Thus, for $h_1 \neq 0$, all orbits are located within an unbounded region of space.

With $h_1 = 0$, we have

$$\left(\frac{d\lambda}{d\tau}\right)^2 = -(1 + \lambda^2)\left(\frac{2fM}{c^3}\lambda - 2c_2\right). \tag{4.12}$$

The right-hand member of this equation will be non-negative provided $\lambda_1 < \frac{c_2c^3}{fM}$: in other words, if $h_1 = 0$, there are also no bounded trajectories. /20
Thus, motion will take place in a bounded region only if $h_1 > 0$.

Now we shall demonstrate a theorem which is of central importance for the theory of the perturbed motion of an artificial satellite of a non-spherical planet.

Theorem. Under the influence of perturbations arising from the irregularity in the shape of a planet, Keplerian elliptical motion is transformed into a motion which takes place in an unbounded portion of space, provided certain initial conditions are present. Regardless of what initial conditions may be present, unperturbed parabolic and hyperbolic forms of motion cannot be transformed into bounded forms of motion under the influence of such perturbations.

Proof. We shall express the constants of the integral of kinetic energy

in terms of initial conditions. Then we will have the following:

$$-h_1c^2 = h = -\frac{1}{c^2}\left(\frac{v_0^2}{2} - U_0\right), \tag{4.13}$$

where v_0 and U_0 represent the velocity of the satellite and the force function at the initial moment of time. If $h < 0$, then, as was demonstrated earlier, the motion will take place in a bounded portion of space. If $h \geq 0$, then all of the orbits will retreat to infinity.

We transform equation (4.13) to the following form:

$$h = -\frac{1}{c^2}\left[\left(\frac{v_0^2}{2} - \frac{fM}{r_0}\right) + \left(\frac{fM}{r_0} - U_0\right)\right]. \tag{4.14}$$

It is obvious that the term

$$H = \frac{v_0^2}{2} - \frac{fM}{r_0} \tag{4.15}$$

characterizes the mechanical energy of unperturbed Keplerian motion. As the initial moment of time, we shall take the moment at which the satellite passes over one of the poles. Then, $x_0 = 0$, $z_0 = r_0$, and formula (4.14) can be written as follows:

$$h = -\frac{1}{c^2}\left[H + \frac{fMc^2}{r_0(r_0^2 + c^2)}\right]. \tag{4.16}$$

If $H < 0$, then the unperturbed motion will be elliptical. If r_0 is chosen /202
sufficiently small, the constant h can be made positive, and then the perturbed motion will take place in an unbounded portion of space, despite the fact that at the initial moment it was elliptical.

If $H \geq 0$, then $h < 0$. Consequently, both the perturbed and the unperturbed motion will be unbounded. The theorem has been proved.

We shall consider only those polar orbits which lie within a bounded portion of space, and disregard all other cases. We should distinguish two cases of bounded motion. In one of these cases the coordinate λ during the process of motion remains enclosed between two boundaries, and in the other case λ is constant.

Let us examine the quadrature which determines μ. Here it is possible to distinguish four cases: a) $\delta > 1$; b) $0 < \delta < 1$; c) $\delta = 1$; and d) $\delta = 0$. In case a) the coordinate μ varies between -1 to +1. In case b) μ varies between $-\delta$ and $+\delta$. In case c) μ varies between -1 to +1. Here motion with a constant

value of μ is also possible. In case d) μ remain invariable.

Combining these four cases with those which were obtained in the analysis of the quadrature for λ, we arrive at the following five types of motion.

Type 1a). This is the case in which the roots of the polynomial $\phi(\lambda)$ are real and distinct, while $\delta > 1$. Here the motion takes place within an elliptical annulus. In the general case the motion will be almost periodic, and will everywhere densely fill the region of possible motion.

Type 2a). In this case the roots of the polynomial $\phi(\lambda)$ are equal and $\delta > 1$. This type includes elliptical orbits

$$\lambda = -\frac{fM}{2h_1c^3}, \quad -1 \leqslant \mu \leqslant 1. \tag{4.17}$$

Type 1b). In this case $\alpha \neq \beta$, while $\delta < 1$. Consequently, the region of possible motion is bounded by two ellipses $\lambda = \alpha$ and $\lambda = \beta$ and by the two branches of the hyperbola $\mu = \delta$. The coordinates of the artificial earth satellite satisfy the inequalities $-\delta \leq \mu \leq \delta$, $\beta \leq \lambda \leq \alpha$. Since $\delta < 1$, /20

$$|\alpha| = \left| -\frac{fM}{2h_1c^3} + \sqrt{\left(\frac{fM}{2h_1c^3}\right)^2 - \delta^2} \right| < 1,$$

and therefore one of the ellipses bounding the region of possible motion lies within the body of the earth. Consequently, in the process of satellite movement it must contact the earth.

Type 1c). Here $-1 \leq \mu \leq 1$, $\beta \leq \lambda \leq \alpha$. In this case a collision with the earth will also take place, since

$$|\alpha| = \left| -\frac{fM}{2h_1c^3} + \sqrt{\left(\frac{fM}{2h_1c^3}\right)^2 - \delta^2} \right| \leqslant 1. \tag{4.18}$$

We should note that rectangular motion along the earths axis of rotation is also possible. As a matter of fact, $\mu = \pm 1$ represents a solution of equation (4.4). From formulas (4.1) we find that

$$x = 0, \quad z = \pm c\lambda.$$

Type 1d). In this case $\mu = 0$, $\beta \leq \lambda \leq \alpha$. From the relationships of (4.1) it is evident that $x = c\sqrt{1 + \lambda^2}$, $z = 0$; i.e., the motion takes place along a straight line lying within the equatorial plane. Since $\alpha = 0$, $c \leq x \leq c\sqrt{1 + \beta^2}$.

The latter three types of motion are of no practical interest in the theory of the motion of artificial earth satellites, since they lead to collision with the earth.

Let us now consider elliptical orbits in somewhat greater detail. For such orbits to exist it is necessary that the conditions of (4.11) be met. From this it follows that the constants of integration must be related as follows:

$$\alpha = \beta = -\delta = -\frac{fM}{2h_1c^3}. \tag{4.19}$$

As we have seen, the coordinate λ is constant, its value being defined by the formula

$$\lambda = -\frac{fM}{2h_1c^3}. \tag{4.20}$$

Let us find an expression for the second coordinate μ. To do this we transform the elliptic integral of (4.2) to normal form /204

$$\int_0^{\mu} \frac{d\mu}{\sqrt{(1-\mu^2)(1-k^2\mu^2)}} = m\tau, \tag{4.21}$$

where

$$k = 1/\delta, \quad m = \delta\sqrt{2h_1}. \tag{4.22}$$

Transforming the elliptic integral, we find that

$$\mu = \mathrm{sn}\ (m\tau,\ k). \tag{4.23}$$

Substituting the values found for λ and μ in formulas (4.1), we find that

$$x = a\ \mathrm{cn}\ m\tau, \quad z = b\ \mathrm{sn}\ m\tau, \tag{4.24}$$

where

$$a = c\sqrt{1 + \frac{fM}{2h_1c^3}}, \quad b = \frac{fM}{2h_1c^3}\cdot c. \tag{4.25}$$

The quantities a and b represent, respectively, the major and the minor semi-axes of the elliptical orbit, whose eccentricity is equal to e. The modulus of the elliptic Jacobi functions and the eccentricity are related as follows:

$$k = \frac{e}{\sqrt{1-c^2}}. \tag{4.26}$$

Since the quantity c is quite small in comparison with the major semi-axis, the eccentricity and the absolute value of k are small quantities. It should be noted that when c = 0 the elliptical orbits in question degenerate into circular orbits.

Now let us find the relationship between the regularizing variable τ and the time t. From (6.47) and (6.61) Chapter III, we know that

$$t - t_0 = \lambda^2 \tau + \int_0^{\tau} \mu^2 \, d\tau, \tag{4.27}$$

where t_0 is the initial moment.

Substituting in equation (4.27) the value of μ obtained from (4.23), we find that

$$t - t_0 = \lambda^2 \tau + \frac{1}{m} \int_0^{s} \frac{s^2 \, ds}{\sqrt{(1-s^2)(1-k^2 s^2)}} \tag{4.28}$$

(here for convenience we have substituted s = sn $m\tau$). This relationship can be represented in the following form: /205

$$t - t_0 = \lambda^2 \tau + \frac{1}{mk^2} \{F(\operatorname{am} m\tau) - E(\operatorname{am} m\tau)\}, \tag{4.29}$$

where F and E represent elliptic integrals of the first and the second type.

From (4.21) it follows that motion along an elliptical orbit is periodic with respect to τ having a period of 4K(k)/m. Using formula (4.29), we find the period with respect to t:

$$T = \frac{4\lambda^2}{m} K(k) + \frac{1}{mk^2} [K(k) - E(k)]. \tag{4.30}$$

Taking advantage of the degree of smallness of the absolute value of k and of the eccentricity of the orbit, we expand (4.30) in a power series of e:

$$T = \frac{2\pi}{n} \left[1 + \frac{9}{8} e^2 + \frac{3}{2} e^4 + \cdots \right], \tag{4.31}$$

where

$$n = \sqrt{\frac{fM}{a^3}}. \tag{4.32}$$

From this it is evident that the period of rotation of the satellite depends not only upon the mass of the earth, but also upon the degree of compression of the planet.

Let us proceed now to the study of the second case of satellite polar orbits, for which $\beta \leq \lambda \leq \alpha$, $-1 \leq \mu \leq 1$. The motion takes place within an elliptical annulus. It is not difficult to determine that the major semi-axis and the eccentricity of the outer elliptical boundary are expressed as follows in terms of the roots α:

$$a_1 = c\sqrt{1+\alpha^2}, \qquad e_1 = \frac{1}{\sqrt{1+\alpha^2}}. \tag{4.33}$$

The corresponding quantities for the inner ellipse are:

$$a_2 = c\sqrt{1+\beta^2}, \qquad e_2 = \frac{1}{\sqrt{1+\beta^2}}. \tag{4.34}$$

The coordinate μ retains its earlier value. The coordinate λ is found from equation (4.5), by transforming the corresponding elliptic integral [136]:

$$\lambda = \frac{A + B\,\mathrm{cn}\,[\omega(\tau-\tau_0), \bar{k}]}{C + D\,\mathrm{cn}\,[\omega(\tau-\tau_0), \bar{k}]}, \tag{4.35}$$

where the following designations are employed: /206

$$\left.\begin{aligned} A &= \beta\sqrt{1+\alpha^2} + \alpha\sqrt{1+\beta^2}, \quad C = \sqrt{1+\beta^2} + \sqrt{1+\alpha^2}, \\ B &= \beta\sqrt{1+\alpha^2} - \alpha\sqrt{1+\beta^2}, \quad D = \sqrt{1+\alpha^2} - \sqrt{1+\beta^2}, \\ \bar{k}^2 &= \frac{1}{2}\left\{1 - \frac{1+\alpha\beta}{\sqrt{(1+\alpha^2)(1+\beta^2)}}\right\}, \\ \omega &= \sqrt[4]{4h_1^2(1+\alpha^2)(1+\beta^2)}. \end{aligned}\right\} \tag{4.36}$$

Substituting in equation (4.1) the values found for the coordinates μ and λ, we find that

$$x = \frac{2N\,\mathrm{dn}\,[\omega(\tau-\tau_0), \bar{k}]}{1-\gamma\,\mathrm{cn}\,[\omega(\tau-\tau_0), \bar{k}]}\,\mathrm{cn}\,(m\tau, k), \tag{4.37}$$

$$z = \frac{\sqrt{1-e_1^2} + \sqrt{1-e_2^2} - \left(\sqrt{1-e_1^2} - \sqrt{1-e_2^2}\right)\mathrm{cn}\,[\omega(\tau-\tau_0), \bar{k}]}{1-\gamma\,\mathrm{cn}\,[\omega(\tau-\tau_0), \bar{k}]} \times N\,\mathrm{sn}\,(m\tau, k),$$

where

$$\left.\begin{aligned} N &= -\frac{a_1 a_2}{a_1 + a_2}, \qquad \gamma = \frac{a_2 - a_1}{a_2 + a_1}, \\ m &= \sqrt{\frac{a_1\sqrt{1-e_1^2} + a_2\sqrt{1-e_2^2}}{2} \cdot \frac{fM}{c^4}}, \\ \omega &= \sqrt{\frac{2a_1 a_2}{a_1\sqrt{1-e_1^2} + a_2\sqrt{1-e_2^2}} \cdot \frac{fM}{c^4}}, \\ k^2 &= \frac{4e_1^2 e_2^2}{\left(e_1\sqrt{1-e_1^2} + e_2\sqrt{1-e_2^2}\right)^2}, \\ \bar{k}^2 &= \frac{1}{2}\left[1 + e_1 e_2 - \sqrt{(1-e_1^2)(1-e_2^2)}\right]. \end{aligned}\right\} \tag{4.38}$$

From the expressions for k and $\bar{k}$, it is evident that in this case the absolute values of the elliptic Jacobi functions will be small quantities. Consequently, it is possible to make use of the expansions of the elliptic functions in series of cosines and sines of multiples of $\tau - \tau_0$, where τ_0 is a certain constant. These expansions will be obtained for the general case in the following chapter.

CHAPTER V

DETERMINING THE INTERMEDIATE ORBITS OF ARTIFICIAL SATELLITES

§ 1. Inversion of Quadratures in the Basic Satellite Case /207

Formulas (6.62) - (6.65), derived in Chapter 3, afford a general solution to the generalized problem of two immobile centers. For convenience we shall adopt the notations

$$\xi = c \operatorname{sh} v, \quad \eta = \cos u, \tag{1.1}$$

where the spheroidal coordinates ξ, η are associated with the rectangular coordinates x, y, z by the following relationships:

$$\left.\begin{aligned} x &= \sqrt{(c^2+\xi^2)(1-\eta^2)}\cos\omega, \\ y &= \sqrt{(c^2+\xi^2)(1-\eta^2)}\sin\omega, \\ z &= \xi\eta, \end{aligned}\right\} \tag{1.2}$$

which are easily found with the help of the transformation formulas (6.39) of Chapter 3. Then, instead of (6.62) - (6.65), we will have the following quadratures for the general solution:

$$\int \frac{d\eta}{\sqrt{\eta^4 - \left(\frac{c_2}{hc^2}+1\right)\eta^2 + \left(\frac{c_2}{hc^2}+\frac{c_1^2}{2hc^2}\right)}} = \sqrt{-2hc^2}(\tau + c_3), \tag{1.3}$$

$$\int \frac{d\xi}{\sqrt{-\xi^4 - \frac{fM}{h}\xi^3 - \left(\frac{c_2}{h}+c^2\right)\xi^2 - \frac{fMc^2}{h}\xi - c^2\left(\frac{c_2}{h}+\frac{c_1^2}{2h}\right)}} = \sqrt{-2h}(\tau + c_4), \tag{1.4}$$

$$\omega = c_1\int \frac{(\xi^2+c^2\eta^2)\,d\tau}{(\xi^2+c^2)(1-\eta^2)} + c_5, \tag{1.5}$$

$$t = \int (\xi^2 + c^2\eta^2)\,d\tau. \tag{1.6}$$

In what follows we shall be concerned with the reduction of these quadratures to a form which is convenient for practical use. We can limit ourselves to the basic satellite case, described in Chapter 3, which is characterized by the fact that the polynomial in quadrature (1.4) has two real and two complex /208

conjoint roots. The elements h, c_1, and c_2, which figure in the solution do not admit of a simple geometrical interpretation, and are therefore not convenient. In their place it is expedient to introduce certain new constants, just as was done earlier in the solution of the Barrar problem (see §§ 4 and 7, Chapter 3). In this connection, we shall make use of the elements proposed by I. Izzak [130]. Basically, we shall follow the procedure of [134][1].

We shall chose the two elements a and e in such a way that the quantities a(1 - e) and a(1 + e) are rea; roots of the polynomial

$$\Phi_2(\xi) = \xi^4 + \frac{fM}{h}\xi^3 + \left(\frac{c_2}{h} + c^2\right)\xi^2 + \frac{fMc^2}{h}\xi + \frac{c^2}{h}\left(c_2 + \frac{c_1^2}{2}\right). \tag{1.7}$$

We should note that throughout the motion the quantity ξ will be included between the two indicated roots $\xi_1 = a(1 - e)$ and $\xi_2 = a(1 + e)$. The two other roots of the polynomial (1.7) in the case in question will be complex-conjoint. We shall represent them in the following form:

$$\xi_3 = \alpha + i\beta, \quad \xi_4 = \alpha - i\beta. \tag{1.8}$$

The elements a and e are analogous to the major semi-axis and the eccentricity of the elliptical orbit, respectively; when c = 0, they coincide. Actually, the last point constitutes their principal advantage.

It is obvious that the elements a and e are associated with the earlier constants as follows:

$$a^4(1-e)^4 + \frac{fM}{h}a^3(1-e)^3 + \left(\frac{c_2}{h} + c^2\right)a^2(1-e)^2 + \frac{fM}{h}ac^2(1-e) + c^2\left(\frac{c_2}{h} + \frac{c_1^2}{2h}\right) = 0, \tag{1.9}$$

and

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$$a^4(1+e)^4 + \frac{fM}{h}a^3(1+e)^3 + \left(\frac{c_2}{h} + c^2\right)a^2(1+e)^2 + \frac{fM}{h}c^2a(1+e) + c^2\left(\frac{c_2}{h} + \frac{c_1^2}{2h}\right) = 0. \tag{1.10}$$

We now introduce a third element in order to make allowance for the second polynomial which figures in the general solution of the problem:

$$\Phi_1(\eta) = \eta^4 - \frac{1}{c^2}\left(\frac{c_2}{h} + c^2\right)\eta^2 + \frac{1}{c^2}\left(\frac{c_2}{h} + \frac{c_1^2}{2h}\right). \tag{1.11}$$

[1]Quasi-Keplerian elements of the orbit were first introduced by Tilhet in the plane problem of two immobile centers.

Let the roots of this polynomial be $\pm s$, $\pm\gamma$. Then the polynomial $\Phi_1(\eta)$ can be represented as follows:

$$\Phi_1(\eta) = (\eta^2 - s^2)\ (\eta^2 - \gamma^2). \tag{1.12}$$

The roots γ and s are related to each other as follows:

$$\gamma = \sqrt{1 - s^2 + \frac{c_2}{hc^2}}. \tag{1.13}$$

Let us take the root s as a new element. Then by $\pm s$ we shall understand those roots of the equation between which the coordinate η varies during the process of motion. If c is set equal to zero, then the root s coincides with the sine of the inclination of the orbit in the two-body problem.

One of the earlier constants, namely c_1, is easily excluded, by representing it in terms of h, c_2, s:

$$c_1^2 = -2hc^2\left[s^4 + \frac{c_2}{hc^2} - \frac{hc^2 + c_2}{hc^2}s^2\right] \tag{1.14}$$

Since the approximating potential coincides with the actual potential with an accuracy on the order of the square of the compression, it is expedient that all quantities which figure in the solution should be calculated with an accuracy on the order of the square of the compression. In order to do this we shall make use of series expansions of the small dimensionless parameter ε, which is defined as follows:

$$\varepsilon = \frac{c}{p},$$

where $p = a(1 - e^2)$ is the analogue of the focal parameter.

This quantity does not exceed 1/30, as follows from the criterion of real motion (1.26), Chapter 4. Therefore, in the series expansion of the quantities necessary for this purpose, we can limit ourselves to terms which contain ε^4. /210

Substituting in equation (1.10) the value of c_1 from (1.14), and solving the system thus obtained for c_2/h and fM/h with accuracy on the order of ε^4 inclusive, we arrive at

$$\frac{c_2}{h} = p\{1 + \varepsilon^2(3 + e^2)(1 - s^2) + \varepsilon^4[4(1 - e^2) - 16s^2 + 4(3 + e^2)s^4]\} + \ldots, \tag{1.15}$$

$$\frac{fM}{h} = -2a\{1 + \varepsilon^2(1 - e^2)(1 - s^2) + \varepsilon^4[(1 - e^2)^2 - (5 - e^2)(1 - e^2)s^2 + 4(1 - e^2)s^4]\} + \ldots \tag{1.16}$$

The values found for h and c_2 we now substitute in (1.14):

$$c_1 = \pm \sqrt{fMp\,(1-s^2)}\left\{1+\varepsilon^2\left[(1+e^2)-\frac{1}{2}(3+e^2)\,s^2\right]+\right.$$
$$+\varepsilon^4\left[-2e^2-\frac{1}{2}(3+4e^2+e^4)\,s^2+\right.$$
$$\left.\left.+\left(\frac{11}{8}+\frac{17}{4}e^2+\frac{3}{8}e^4\right)s^4\right]\right\}+\ldots \qquad (1.17)$$

In this formula the plus sign shall be resumed to refer to orbits with direct motion, and the minus sign to orbits with retrograde motion. In other words, c_1 is positive when the inclination of the orbit is less than 90°, and negative when the inclination exceeds 90°.

In what follows we shall also require expressions for the roots ξ_3, ξ_4 and γ, in terms of the Izzak elements. In order to do this, we shall substitute expressions for c_1, c_2 and h in the polynomial of (1.8) and, setting the polynomial equal to zero, find the following expressions for α and β:

$$\alpha = p\{\varepsilon^2\,(1-s^2)+\varepsilon^4\,[1-e^2-(5-e^2)\,s^2+4s^2]\}, \qquad (1.18)$$

$$\beta^2 = p\varepsilon^2\,[s^2-\varepsilon^2\,(1-6s^2+5s^4)]. \qquad (1.19)$$

In the basic satellite case, the roots ξ_3 and ξ_4 must be complex: i.e., as is evident from (1.19), the following condition must be observed:

$$s^2-\varepsilon^2\,(1-6s^2+5s^4)>0 \qquad (1.20)$$

or /211

$$|s|>\varepsilon\,(1-3\varepsilon^2). \qquad (1.21)$$

But $\varepsilon < 1/30$, and therefore, for the existence of complex roots it is sufficient that $|s| > 0.033$. Since the quantity s characterizes the inclination of the orbit, the latter inequality means that we must exclude from consideration all orbits which are nearly equatorial. The equatorial orbit, in this case requires a special investigation.

Making due allowance for equation (1.15), we expand equation (1.13) in power series of ε:

$$\gamma = \frac{1}{\varepsilon\sqrt{1-e^2}}\{1+2\varepsilon^2\,(1-s^2)-2\varepsilon^4\,[e^2+2s^2-(2+e^2)\,s^4]\}+\ldots \qquad (1.22)$$

Now the integral (1.3) can be represented in the form:

$$\int \frac{d\left(\frac{\eta}{s}\right)}{\sqrt{\left[1-\left(\frac{\eta}{s}\right)^2\right]\left[1-k^2\left(\frac{\eta}{s}\right)^2\right]}} = \sigma(\tau + c_3), \tag{1.23}$$

where we have employed the designations

$$\sigma = c\gamma\sqrt{-2h}, \tag{1.24}$$

$$k = \frac{s}{\gamma}. \tag{1.25}$$

Inverting the elliptic quadrature (1.23) we find that

$$\eta = s\,\mathrm{sn}\,[\sigma(\tau + c_3),\ k]. \tag{1.26}$$

With the help of formulas (1.15), (1.16) and (1.22), the quantities k and σ can be represented in the form of power series of ε. As before we limit ourselves to their first terms:

$$k^2 = \varepsilon^2(1-e^2)\,s^2\,[1-4\varepsilon^2(1-s^2)] + \ldots, \tag{1.27}$$

$$\sigma = \sqrt{fMp}\left\{1+\frac{1}{2}\varepsilon^2(3+e^2)(1-s^2) - \varepsilon^4\left[\frac{1}{8}(9+6e^2+e^4) + \right.\right.$$
$$\left.\left. + \frac{1}{4}(1+14e^2+e^4)s^2 - \frac{1}{8}(11+34e^2+3e^4)\,s^4\right]\right\} + \ldots \tag{1.28}$$

As is apparent from (1.27), the absolute value of k is on the same order as that of ε.

Inversion of the second quadrature of (1.4) does not represent any difficulty; it is written as follows: /212

$$\int \frac{d\xi}{\sqrt{(\xi_2-\xi)(\xi-\xi_1)[(\xi-\alpha)^2+\beta^2]}} = \sqrt{-2h}\,(\tau + c_4). \tag{1.29}$$

With the help of the tables of [141] we find that

$$\xi = \frac{\bar{p}\{1+\varkappa\,\mathrm{cn}\,[\sigma_1(\tau+c_4),\ k_1]\}}{1+\bar{e}\,\mathrm{cn}\,[\sigma_1(\tau+c_4),\ k_1]}, \tag{1.30}$$

where

$$\bar{p} = \frac{a\,[\bar{\alpha}(1-e) + \bar{\beta}(1+e)]}{\bar{\alpha}+\bar{\beta}}, \tag{1.31}$$

$$\varkappa = \frac{\bar{\alpha}(1-e) - \bar{\beta}(1+e)}{\bar{\alpha}(1-e) + \bar{\beta}(1+e)}, \tag{1.32}$$

$$\bar{e} = \frac{\bar{\alpha}-\bar{\beta}}{\bar{\alpha}+\bar{\beta}}, \tag{1.33}$$

in which

$$\bar{\alpha} = \alpha^2 + \beta^2 - 2\alpha a\,(1+e) + a^2\,(1+e)^2, \tag{1.34}$$

$$\bar{\beta} = \alpha^2 + \beta^2 - 2\alpha a\,(1-e) + a^2\,(1-e)^2. \tag{1.35}$$

The constant σ_1 and the modulus of the Jacobian elliptic function k_1 are related as follows:

$$\sigma_1 = \sqrt{-2h\bar{\alpha}\bar{\beta}}, \tag{1.36}$$

$$k_1 = \frac{a^2\,[4e^2 - (\bar{\alpha}-\bar{\beta})^2]}{4\bar{\alpha}\bar{\beta}}. \tag{1.37}$$

The coefficients and the auxiliary constants introduced in formulas (1.30) - (1.37) can also be represented in the form of series:

$$\bar{e} = e\,\{1 + \varepsilon^2\,(1-e^2)\,(1-2s^2) + \varepsilon^4\,(1-e^2)\,[(3-16s^2 + 14s^4) - 2e^2\,(1-s^2)^2]\} + \ldots, \tag{1.38}$$

$$\varkappa = e\varepsilon^2\,\{(1-2s^2) + \varepsilon^2\,[(3-16s^2+14s^4) - e^2\,(1-2s^4)]\} + \ldots, \tag{1.39}$$

$$\bar{p} = a\,(1 - e\bar{e}), \tag{1.40}$$

and

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$$k_1^2 = e^2\varepsilon^2\,[s^2 - \varepsilon^2\,(1 - 10s^2 + 11s^4 + e^2 s^4)] + \ldots, \tag{1.41}$$

$$\sigma_1 = \sqrt{fMp}\left\{1 - \varepsilon^2\left(\frac{3-e^2}{2} - 2s^2\right) - \frac{\varepsilon^4}{8}\,[(9 + 2e^2 + e^4) - (72 + 40e^2)\,s^2 + (64 + 48e^2)\,s^4]\right\} + \ldots. \tag{1.42}$$

As is evident from formula (1.41), the modulus of the elliptic function in (1.30) is quite small, since it is proportional to ε.

From formulas (3.22) Chapter I and (1.1) of the present paragraph, it

follows that if ξ = const, the satellite necessarily travels on the ellipsoid:

$$\frac{x^2+y^2}{\xi^2+c^2}+\frac{z^2}{\xi^2}=1. \tag{1.43}$$

However, according to formula (1.30),

$$a(1-e)\leq \xi \leq a(1+e).$$

Thus we see that the motion of the satellite takes place within the ellipsoidal layer. The major and the minor semi-axes of the outer boundary ellipsoid are equal to $\sqrt{a^2(1+e)^2+c^2}$ and $a(1+e)$, respectively, and those of the inner ellipsoid are equal to $\sqrt{a^2(1-e)^2+c^2}$ and $a(1-e)$. The eccentricities of the inner and the outer boundary ellipsoids have the values $\varepsilon(1+e)^2[1+\varepsilon^2(1+e)^2]^{-1/2}$, $\varepsilon(1-e)^2[1+\varepsilon^2(1-e)^2]^{-1/2}$. It follows from this that the eccentricities of the ellipsoids diminish when their major semi-axes are increased. The greatest difference between the major and the minor semi-axes of the boundary ellipsoids amounts to 4 km.

The other boundary surface of the region of possible motion is the single-sheet hyperboloid of rotation $\eta = s$, which generates from the ellipsoidal layer the toroid within which the motion takes place. The semi-axes of the hyperboloid and its eccentricity are equal to

$$c\sqrt{1-s^2},\quad cs,\quad \frac{1}{\sqrt{1-s^2}},$$

respectively.

§ 2. Expansion of Satellite Coordinates in Series

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Formulas (1.26) and (1.30) represent ellipsoidal coordinates in terms of Jacobian elliptic functions. However, in a number of practical situations, particularly in the construction of a precise analytical theory of the perturbed motion of satellites [137-139], it is more convenient to make use of expansions in power series of the moduli of the elliptic integrals (1.3), and (1.4). Such expansions may be obtained in a number of different ways. The early terms of expansions can be calculated quite easily with the help of the Landen transformation [141, 141], which leads to a rapid decrease in the moduli of the elliptic integrals (see § 3). Good results can also be obtained by the expansion of coordinates in series of theta functions. Also, by the use of trigonometric series for the elliptic functions. The latter method has been used by I. Izzak [130] and Ye. P. Aksenov [134]. Following these writers, we introduce the auxiliary variables ϕ and v, defined as follows:

$$\varphi = \operatorname{am}\ \sigma(\tau + c_3), \tag{2.1}$$

$$v = \operatorname{am} \sigma_1 (\tau + c_4). \tag{2.2}$$

Expanding these functions in power series of the moduli k and k_1, we find that

$$\sigma(\tau + c_3) = \left(1 + \frac{k^2}{4} + \frac{9}{64} k^4\right) \varphi - \frac{k^2}{8}\left(1 + \frac{3}{4} k^2\right) \sin 2\varphi + \frac{3}{256} k^4 \sin 4\varphi + \ldots, \tag{2.3}$$

$$\sigma_1(\tau + c_4) = \left(1 + \frac{1}{4} k_1^2 + \frac{9}{64} k_1^4\right) v - \frac{k_1^2}{8}\left(1 + \frac{3}{4} k_1^2\right) \sin 2v + \frac{3}{256} k_1^4 \sin 4v + \omega + \ldots, \tag{2.4}$$

where ω is a new constant related to c_3 and c_4 as follows:

$$\omega = \sigma_1 (c_3 - c_4). \tag{2.5}$$

Excluding regularized time τ from equations (2.3) and (2.4), we arrive at a relationship between the quantities v and ϕ: /215

$$\varphi - \left(\frac{k^2}{8} + \frac{k^4}{16}\right) \sin 2\varphi + \frac{3}{256} k^4 \sin 4\varphi = \omega + (1 + \nu)\left[v - \right.$$
$$\left. - \frac{1}{16}(2k_1^2 + k_1^4) \sin 2v + \frac{3}{256} k_1^2 \sin 4v\right] + \ldots, \tag{2.6}$$

in which ν is defined as follows:

$$\nu = -1 + \frac{64 + 16k_1^2 + 9k_1^4}{64 + 16k^2 + 9k^4}. \tag{2.7}$$

From this, with the help of (1.27) and (1.41), we find that

$$\nu = \frac{\varepsilon^2}{4}(12 - 15s^2) + \frac{\varepsilon^4}{64}[(288 - 1296s^2 + 1035s^4) - e^2(144 + 288s^2 - 510s^4)] + \ldots \tag{2.8}$$

Solving equation (2.6) for the variable ϕ, we arrive at

$$\varphi = u + \left[-\frac{1}{8}\varepsilon^2 e^2 s^2 + \frac{1}{8}\varepsilon^4 e^2\left(1 - 13s^2 + \frac{59}{4}s^4 + \right.\right.$$
$$\left.\left. + \frac{1}{2} e^2 s^4\right)\right] \sin 2v + \left\{\frac{\varepsilon^2}{8}(1 - e^2)s^2 - \frac{\varepsilon^2}{16} s^2 [(8 - 9s^2) - \right.$$
$$\left. - e^2(8 - 10s^2) - e^4 s^2]\right\} \sin 2u + \frac{3}{256}\varepsilon^4 e^4 s^4 \sin 4v +$$
$$+ \frac{1}{256}\varepsilon^4 (1 - e^2)^2 s^4 \sin 4u + \frac{\varepsilon^4}{64} e^2 (1 - e^2) s^4 \sin 2(u - v) -$$

$$-\frac{\varepsilon^4}{64}e^2(1-e^2)s^4\sin 2(u+v)+\ldots, \tag{2.9}$$

in which we have adopted the designation:

$$u = (1 + \nu)\, v + \omega. \tag{2.10}$$

Let us proceed now to the expansion of the geometric radius-vector of the satellite in trigonometric series, for which, from formula (1.2), we know that

$$r=\sqrt{(c^2+\xi^2)(1-\eta^2)+\xi^2\eta^2}. \tag{2.11}$$

The expansion of the right-hand member of this formula in power series of c/ξ (with accuracy on the order of the fourth degree of this quantity) is as follows:

$$r=\xi\left[1+\frac{1}{2}\left(\frac{c}{\xi}\right)^2(1-\eta^2)-\frac{1}{8}\left(\frac{c}{\xi}\right)^4(1-\eta^2)^2\right]. \tag{2.12}$$

Taking into consideration formulas (2.1) and (2.2), the relationships of (1.26) and (1.30) can be represented as follows: /216

$$\eta = s\sin\varphi, \tag{2.13}$$

$$\xi=\frac{\bar{p}(1+\varkappa\cos v)}{1+\bar{e}\cos v}. \tag{2.14}$$

Substituting these expressions in (2.12) and retaining only those terms which contain ε up to a certain power, we arrive at

$$\begin{aligned} r=\frac{\bar{p}}{1+\bar{e}\cos v}\{&p_{0,0}+p_{0,1}\cos v+p_{0,2}\cos 2v+p_{0,3}\cos 3v+\\ &+p_{2,0}\cos 2\varphi+p_{2,-1}\cos(2\varphi-v)+p_{2,1}\cos(2\varphi+v)+\\ &+p_{2,2}\cos(2\varphi+2v)+p_{2,-2}\cos(2\varphi-2v)\}, \end{aligned} \tag{2.15}$$

where

$$\left.\begin{aligned} p_{0,0}&=1+\frac{\varepsilon^2}{8}[(4-2s^2)+e^2(2-s^2)]-\\ &-\frac{1}{8}\varepsilon^4\left[\left(1-s^2+\frac{3}{8}s^4\right)-e^2\left(5-17s^2+\frac{55}{8}s^4\right)+\right.\\ &\left.+e^4\left(\frac{3}{8}-\frac{3}{8}s^2+\frac{9}{64}s^4\right)\right],\\ p_{0,1}&=e\left\{\left(2-\frac{5}{2}s^2\right)\varepsilon^2+\frac{\varepsilon^4}{8}\left[\left(24-134s^2+\frac{229}{2}s^4\right)-\right.\right.\\ &\left.\left.-e^2\left(6+\frac{19}{2}s^2-\frac{159}{8}s^4\right)\right]\right\}, \end{aligned}\right| \tag{}$$

$$
\left.\begin{aligned}
p_{0,2} &= e^2\left\{\frac{\varepsilon^2}{8}(2-s^2)-\frac{\varepsilon^4}{8}\left[\left(3-3s^2+\frac{9}{8}s^4\right)+\right.\right.\\
&\qquad\left.\left.+e^2\left(\frac{1}{2}-\frac{1}{2}s^2+\frac{3}{16}s^4\right)\right]\right\},\\
p_{0,3} &= -\frac{\varepsilon^4}{8}e^3\left(2-\frac{7}{2}s^2+\frac{11}{8}s^4\right),\\
p_{2,0} &= \frac{\varepsilon^2}{8}(2+e^2)s^2-\frac{\varepsilon^4}{8}\left[\left(s^2-\frac{1}{2}s^4\right)-\right.\\
&\qquad\left.-e^2\left(s^2-\frac{13}{2}s^4\right)+e^4\left(\frac{3}{8}s^2-\frac{3}{16}s^4\right)\right],\\
p_{2,-1} &= p_{2,1} = es^2\left\{\frac{\varepsilon^2}{4}-\frac{\varepsilon^4}{16}\left[2(1+s^2)+\right.\right.\\
&\qquad\left.\left.+e^2\left(\frac{1}{2}+\frac{7}{2}s^2\right)\right]\right\},\\
p_{2,2} &= p_{2,-2} = e^2s^2\left\{\frac{\varepsilon^2}{16}-\frac{\varepsilon^4}{16}\left[\left(3-\frac{3}{2}s^2\right)+\right.\right.\\
&\qquad\left.\left.+e^2\left(\frac{1}{2}-\frac{1}{4}s^2\right)\right]\right\}.
\end{aligned}\right\} \qquad (2.16)
$$

In analogous fashion we arrive at the expansion for z: /217

$$
z = \frac{\bar{p}s}{1+e\cos v}\left[\sin\varphi + q_{1,-1}\sin(\varphi - v) + q_{1,1}\sin(\varphi + v)\right], \qquad (2.17)
$$

in which

$$
q_{1,1} = q_{1,-1} = \frac{e\varepsilon^2}{2}\left\{(1-2s^2)+\varepsilon^2\left[(3-16s^2+14s^4)-e^2(1-2s^4)\right]\right\}. \qquad (2.18)
$$

An expression for the longitude w in terms of the variables ϕ and v can be found from formula (1.5). Expanding the function under the integral in (1.5) in power series of c/ξ, we arrive at:

$$
w = c_1\int\left[\frac{1}{1-\eta^2}-\left(\frac{c}{\xi}\right)^2+\left(\frac{c}{\xi}\right)^4\right]d\tau + c_5. \qquad (2.19)
$$

The first of the integrals of (2.19) we transform to the following form, with the help of (2.1):

$$
\int\frac{d\varphi}{\sigma(1-s^2\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}. \qquad (2.20)
$$

Expanding in power series the function under the integral in (2.20), we find that

$$
\int\frac{d\tau}{1-\eta^2} = \frac{1}{\sigma}\left[\int\frac{d\tau}{1-s^2\sin^2\varphi}+\frac{k^2}{2}\int\frac{\sin^2\varphi\,d\varphi}{1-s^2\sin^2\varphi}+\frac{3}{8}k^4\int\frac{\sin^4\varphi\,d\varphi}{1-s^2\sin^2\varphi}\right]. \qquad (2.21)
$$

Integrating, we arrive at

$$\int \frac{d\tau}{1-\eta^2} = \frac{1}{\sigma\sqrt{1-s^2}}\left\{\left(1+\frac{k^2}{2s^2}+\frac{3k^4}{8s^4}\right)\arctan\left(\sqrt{1-s^2}\,\mathrm{tg}\,\varphi\right) - \right.$$
$$\left. -\left[\frac{k^2}{2s^2}+\frac{3k^4}{16s^4}(2+s^2)\right]\sqrt{1-s^2}\,\varphi+\frac{3k^4}{32s^2}\sqrt{1-s^2}\sin 2\varphi\right\}. \qquad (2.22)$$

The second integral of (2.19) is calculated with the help of the following substitution

$$d\tau = \frac{dv}{\sigma_1\sqrt{1-k_1^2\sin^2 v}}. \qquad (2.23)$$

Substituting in (2.19) the values of the integrals found, we finally arrive at the following: /218

$$w = \pm\left[\arctan\left(\sqrt{1-s^2}\tan\varphi\right)+c_0\varphi+c_2\sin 2\varphi+\bar{c}_0 v+\right.$$
$$\left.+\bar{c}_1\sin v+\bar{c}_2\sin 2v+\bar{c}_3\sin 3v+c_5\right], \qquad (2.24)$$

where the coefficients c_i, $\bar{c}_i$ are defined as follows:

$$\left.\begin{aligned}
c_0 &= -\frac{\varepsilon^2}{2}(1-e^2)\sqrt{1-s^2}\times \\
&\quad\times\left\{1-\frac{\varepsilon^2}{8}[(30-35s^2)+(2+3s^2)e^2]\right\}, \\
c_2 &= \frac{3}{32}\varepsilon^4(1-e^2)^2 s^2\sqrt{1-s^2}, \\
\bar{c}_0 &= -\frac{\varepsilon^2}{2}(2+e^2)\left\{1+\frac{\varepsilon^2}{8(2+e^2)}[(24-56s^2)-\right. \\
&\quad\left.-e^2(4+64s^2)-e^4(2+3s^2)]\right\}\sqrt{1-s^2}, \\
\bar{c}_1 &= -2\varepsilon^2\left\{1+\frac{\varepsilon^2}{8}[(4-28s^2)-\right. \\
&\quad\left.-(6+7s^2)e^2]\right\}e\sqrt{1-s^2}, \\
\bar{c}_3 &= \frac{\varepsilon^4 e^3}{4}\sqrt{1-s^2}(2-s^2), \\
\bar{c}_2 &= -\frac{\varepsilon^2 e^2}{4}\sqrt{1-s^2}\left\{1-\frac{\varepsilon^2}{2}[11+e^2(1+s^2)]\right\}.
\end{aligned}\right\} \qquad (2.25)$$

In the formula (2.24) the plus sign corresponds to $i < 90°$, while the minus sign corresponds to $i > 90°$. In other words, the plus sign corresponds to orbits with an inclination less than 90°, while a minus sign corresponds to those with an inclination greater than 90°.

These are not the only expressions which may be found for the coordinates.

For example, it may be more useful to employ expansions in which, instead of the variable ϕ, we have the variable u, which is defined by (2.10). These expansions have the following form:

$$r = \frac{\bar{p}}{1+\bar{e}\cos v}\{a_{0,0} + a_{0,1}\cos v + a_{0,2}\cos 2v + a_{0,3}\cos 3v + \\ + a_{2,0}\cos 2u + a_{2,1}\cos(2u+v) + a_{2,-1}\cos(2u-v) + \\ + a_{2,-2}\cos(2u-2v) + a_{2,2}\cos(2u+2v)\}, \tag{2.26}$$

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$$z = \frac{\bar{p}s}{1+\bar{e}\cos v}\{b_{1,0}\sin u + b_{3,0}\sin 3u + \\ + b_{1,-1}\sin(u-v) + b_{1,1}\sin(u+v) + \\ + b_{1,-2}\sin(u-2v) + b_{1,2}\sin(u+2v) + \ldots\}, \tag{2.27}$$

$$\omega = \pm[\bar{\omega} + \mu u + c_{0,1}\sin v + c_{0,2}\sin 2v + c_{0,3}\sin 3v + c_{2,0}\sin 2u] + \Omega, \tag{2.28}$$

in which the coefficients of expansions (2.26) - (2.28) are calculated by the following formulas:

$$\left.\begin{aligned}
a_{0,0} &= 1 + \frac{\varepsilon^2}{8}[(4-2s^2) + e^2(2-s^2)] - \\
&- \frac{\varepsilon^4}{8}\left[\left(1 - s^2 + \frac{5}{8}s^4\right) - e^2(5 - 17s^2 + 7s^4) + \right. \\
&\left. + e^4\left(\frac{3}{8} - \frac{3}{8}s^2 + \frac{1}{64}s^4\right)\right], \\
a_{0,1} &= e\left\{\varepsilon^2\left(2 - \frac{5}{2}s^2\right) + \frac{\varepsilon^4}{8}\left[(24 - 134s^2 + \right.\right. \\
&\left.\left. + 114s^4) - e^2\left(6 + \frac{19}{2}s^2 - \frac{163}{8}s^4\right)\right]\right\}, \\
a_{0,2} &= e^2\left\{\frac{\varepsilon^2}{8}(2-s^2) - \frac{\varepsilon^4}{8}\left[\left(3 - 3s^2 + \frac{5}{4}s^4\right) + \right.\right. \\
&\left.\left. + e^2\left(\frac{1}{2} - \frac{1}{2}s^2 + \frac{1}{16}s^4\right)\right]\right\}, \\
a_{0,3} &= -\frac{\varepsilon^4 e^3}{8}\left(2 - \frac{7}{2}s^2 + \frac{11}{8}s^4\right), \\
a_{2,0} &= s^2\left\{\frac{\varepsilon^2}{8}(2+e^2) - \frac{\varepsilon^4}{8}\left[\left(1 - \frac{1}{2}s^2\right) - \right.\right. \\
&\left.\left. - e^2\left(1 - \frac{13}{2}s^2\right) + e^4\left(\frac{3}{8} - \frac{3}{16}s^2\right)\right]\right\}, \\
a_{2,-1} &= es^2\varepsilon^2\left\{\frac{1}{4} - \frac{\varepsilon^2}{16}\left[2(1+s^2) + \right.\right. \\
&\left.\left. + e^2\left(\frac{1}{2} + 3s^2\right)\right]\right\}, \\
a_{2,1} &= es^2\varepsilon^2\left\{\frac{1}{4} - \frac{\varepsilon^2}{16}\left[2(1+s^2) + \right.\right.
\end{aligned}\right\} \tag{2.29}$$

$$+e^2\left(\frac{1}{2}+4s^2\right)\Big]\Big\},\Big\}$$

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$$\left.\begin{aligned}
a_{2,-2} &= e^2s^2\varepsilon^2\left\{\frac{1}{16}-\frac{\varepsilon^2}{16}\left[(3-2s^2)+\right.\right.\\
&\qquad\left.\left.+e^2\left(\frac{1}{2}-\frac{1}{2}s^2\right)\right]\right\},\\
a_{2,2} &= e^2s^2\varepsilon^2\left\{\frac{1}{16}-\frac{\varepsilon^2}{16}\left[(3-s^2)+\frac{1}{2}e^2\right]\right\};\\
b_{1,0} &= 1+\frac{\varepsilon^2}{16}(1-e^2)s^2-\frac{\varepsilon^4s^2}{256}[(64-71s^2)-\\
&\qquad-e^2(64-78s^2)-6e^4s^2],\\
b_{3,0} &= s^2\varepsilon^2\left\{\frac{1}{16}(1-e^2)-\frac{\varepsilon^2}{32}[(8-9s^2)-\right.\\
&\qquad\left.-e^2(8-10s^2)-e^4s^2]\right\},\\
b_{1,-1} &= \frac{\varepsilon^2e}{2}(1-2s^2)+\frac{\varepsilon^4e}{32}[(48-255s^2+222s^4)-\\
&\qquad-e^2(16-32s^4)],\\
b_{1,1} &= \frac{\varepsilon^2e}{2}(1-2s^2)+\frac{\varepsilon^4e}{32}[(48-255s^2+222s^4)-\\
&\qquad-e^2(16+2s^2-36s^4)],\\
b_{1,2} &= -b_{1,-2}=-\frac{\varepsilon^2e^2s^2}{16}+\\
&\qquad+\frac{\varepsilon^4e^2}{16}\left(1-13s^2+\frac{235}{16}s^4+\frac{9}{16}e^2s^4\right);
\end{aligned}\right\}\qquad(2.29)$$

$$\left.\begin{aligned}
\mu &= \left\{-\frac{3}{2}+\frac{\varepsilon^2}{16}[(54-39s^2)+72e^2s^2]\right\}\times\\
&\qquad\times\sqrt{1-s^2}\,\varepsilon^2,\\
c_{0,1} &= -\left\{2+\frac{\varepsilon^2}{4}[(4-28s^2)-(6+7s^2)e^2]\right\}\times\\
&\qquad\times e\sqrt{1-s^2}\,\varepsilon^2,\\
c_{0,2} &= -\left\{\frac{1}{4}-\frac{\varepsilon^2}{16}[(22+s^2)+(2+s^2)e^2]\right\}\times\\
&\qquad\times e^2\sqrt{1-s^2}\,\varepsilon^2,\\
c_{0,3} &= \frac{\varepsilon^4}{4}(2-s^2)e^3\sqrt{1-s^2},\\
c_{2,0} &= \frac{\varepsilon^4}{32}(1-e^2)^2s^2\sqrt{1-s^2},\\
\tan\bar{\omega} &= \sqrt{1-s^2}\tan\varphi,
\end{aligned}\right\}\qquad(2.30)$$

$$(2.31)$$

where, instead of ϕ, it is necessary to substitute its value from (2.9).

§ 3. A Second Means of Approximate Representation of Coordinates

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Let us consider the elliptic integral

$$\int_0^{\psi}\frac{d\psi}{\sqrt{1-\varkappa^2\sin^2\psi}}=u,\qquad(3.1)$$

whose inversion gives

$$\psi = \text{am } (u, \kappa). \tag{3.2}$$

In our particular problem, as was indicated earlier, the moduli of the elliptic integrals are quantities on the order of $\varepsilon \approx 0.03$. For practical purposes in working formulas, it is sufficient to retain those terms which contain powers of ε no higher than the fourth. With this in mind, we transform the integral (3.1) and its inversion, substituting

$$\sin (2\psi - \psi_1) = \varkappa_1 \sin \psi_1, \tag{3.3}$$

where

$$\varkappa_1 = \frac{\varkappa^2}{(1+\varkappa')^2}, \tag{3.4}$$

in which

$$\varkappa'^2 = 1 - \varkappa^2. \tag{3.5}$$

Instead of (3.1), we arrive at

$$u = \frac{1+\varkappa_1}{2} \int_0^{\psi_1} \frac{d\psi_1}{\sqrt{1-\varkappa_1^2 \sin^2 \psi_1}}. \tag{3.6}$$

In problems of the motion of artificial earth satellites, κ_1 will always be on the order of ε^2.

Now we apply the Landen transformation once more:

$$\sin (2\psi_1 - \psi_2) = \varkappa_2 \sin \psi_2, \tag{3.7}$$

where

$$\varkappa_2 = \frac{\varkappa_1^2}{(1+\varkappa_1')^2}, \tag{3.8}$$

while $\kappa_1'^2 = 1 - \kappa_1^2$. Then, /222

$$u = \frac{(1+\varkappa_1)(1+\varkappa_2)}{4} \int_0^{\psi_2} \frac{d\psi_2}{\sqrt{1-\varkappa_2^2 \sin^2 \psi_2}}. \tag{3.9}$$

But κ_2 is on the order of ε^4, and therefore the quantity κ_2^2 can be neglected. Then (3.9) gives

$$\psi_2 = \frac{4u}{(1+\kappa_1)(1+\kappa_2)}. \tag{3.10}$$

From (3.1), (3.3) and (3.7) we arrive at the following expansion of ψ in power series of κ_1 and κ_2:

$$\psi = \frac{1}{4}\psi_2 + \frac{\kappa_1}{2}\sin\frac{\psi_2}{2} + \frac{\kappa_2}{4}\sin\psi_2, \tag{3.11}$$

in which terms will retain to the fourth power inclusively relative to ε.

Assuming

$$\bar{\psi} = \frac{\psi_2}{4} \tag{3.12}$$

and analyzing functions sin ψ and cos ψ:

$$\left.\begin{aligned} \sin\psi &= \sin\operatorname{am} u = \operatorname{sn} u, \\ \cos\psi &= \cos\operatorname{am} u = \operatorname{cn} u, \end{aligned}\right\} \tag{3.13}$$

we substitute into the subsequent formulas in place of ψ the expansion of (3.11). Limiting ourselves to terms of order not higher than ε^4, we find

$$\operatorname{sn} u = \left(1 + \frac{\kappa_1}{4} - \frac{\kappa_1^2}{16}\right)\sin\bar{\psi} + \left(\frac{\kappa_1}{4} + \frac{\kappa_2}{8} - \frac{\kappa_1^2}{32}\right)\sin 3\bar{\psi} + \left(\frac{\kappa_2}{8} + \frac{\kappa_1^2}{32}\right)\sin 5\bar{\psi} + \ldots, \tag{3.14}$$

$$\operatorname{cn} u = \left(1 - \frac{\kappa_1}{4} - \frac{\kappa_1^2}{16}\right)\cos\bar{\psi} + \left(\frac{\kappa_1}{4} + \frac{\kappa_1^2}{32} - \frac{\kappa_2}{8}\right)\cos 3\bar{\psi} + \left(\frac{\kappa_2}{8} + \frac{\kappa_1^2}{32}\right)\cos 5\bar{\psi}. \tag{3.15}$$

Let us now go over to inversion of quadrature (1.23). Introducing the symbol

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$$u_1 = \frac{\sigma(\tau + c_3)}{(1+k^{(1)})(1+k^{(2)})} \tag{3.16}$$

and keeping in mind (3.14), we produce

$$\eta = s\left\{\left(1+\frac{k^{(1)}}{4}-\frac{k^{(1)^2}}{16}\right)\sin u_1+\left(\frac{k^{(1)}}{4}+\frac{k^{(2)}}{8}-\frac{k^{(1)^2}}{32}\right)\sin 3u_1+\left(\frac{k^{(2)}}{8}+\frac{k^{(1)^2}}{32}\right)\sin 5u_1\right\}+\ldots, \tag{3.17}$$

where $k^{(1)}$ and $k^{(2)}$ are determined by formulas (3.4) and (3.8), if we substitute k into the latter.

Similarly from (1.30) and (2.15), we produce the first terms of the series representing ξ:

$$\xi = \frac{p\varkappa}{\bar{e}}+\frac{\bar{e}-\varkappa}{\bar{e}}\cdot\frac{1}{1+\tilde{e}\cos u_2}\left\{1-\frac{a^{(1)}\cos 3u_2+a^{(2)}\cos 5u_2}{1+\tilde{e}\cos u_2}+\frac{\bar{e}^2k_1(1+\cos 6u_2)}{32(1+\tilde{e}\cos u_2)^2}\right\}, \tag{3.18}$$

where

$$\left.\begin{aligned}\tilde{e}&=\bar{e}\left(1-\frac{1}{4}k_1-\frac{1}{16}k_1^2\right), \quad u_2=\frac{\sigma_1(\tau+c_4)}{(1+k_1^{(1)})(1+k_1^{(2)})},\\ a^{(1)}&=\frac{k_1^{(1)}}{4}+\frac{k_1^{(1)^2}}{32}-\frac{k_1^{(2)}}{8}, \quad a^{(2)}=\frac{k_1^{(1)^2}}{32}+\frac{k_1^{(2)}}{8}.\end{aligned}\right\} \tag{3.19}$$

§ 4. Time Equation

The relationship between regularized time τ and time t is given by formula (1.6), which can be written in the form

$$t-\bar{t}_0 = c^2\left[\int\left(\frac{\xi}{c}\right)^2 d\tau+\int\eta^2 d\tau\right]. \tag{4.1}$$

Similarly to the preceeding, the second of the integrals in (4.1) is calculated by expansion into a series and introduction of a new independent variable ϕ. This integral, with an accuracy to qualities on the order of ε^4, is equal to /224

$$\int c^2\eta^2 d\tau = D_0\varphi+D_2\sin 2\varphi+D_4\sin 4\varphi+\ldots, \tag{4.2}$$

where

$$\left.\begin{aligned}D_0&=\varepsilon^2s^2\sqrt{\frac{a^3(1-e^2)^3}{fM}}\left\{\frac{1}{2}-\frac{\varepsilon^2}{16}[(12-15s^2)+(4-s^2)e^2]\right\},\\ D_2&=-\varepsilon^2s^2\sqrt{\frac{a^3(1-e^2)^3}{fM}}\left\{\frac{1}{4}-\frac{\varepsilon^2}{8}[(3-4s^2)+e^2]\right\},\\ D_4&=\frac{\varepsilon^4s^4}{64}\sqrt{\frac{a^3(1-e^2)^3}{fM}}(1-e^2).\end{aligned}\right\} \tag{4.3}$$

The approximate calculation of the first integral gives us

$$\int \xi^2 d\tau = \frac{\bar{p}^2}{\sigma_1}\left\{\frac{2}{(1-\bar{e}^2)^{3/2}}[A_0+(1-\bar{e}^2)A_1]\arctan\sqrt{\frac{1-\bar{e}}{1+\bar{e}}}\times\right.$$
$$\left.\times \operatorname{tg}\frac{v}{2}-\frac{A_0\bar{e}}{1-\bar{e}^2}\cdot\frac{\sin v}{1+\bar{e}\cos v}+B_0 v+B_1\sin v+B_2\sin 2v\right\}, \qquad (4.4)$$

where

$$\left.\begin{aligned}
A_0 &= 1+\varepsilon^2\left[-\left(2-\frac{7}{2}s^2\right)+\frac{1}{2}e^2s^2\right]-\varepsilon^4\left[\left(\frac{5}{2}-17s^2+\right.\right.\\
&\qquad\left.+\frac{113}{8}s^4\right)+\left(\frac{1}{2}-11s^2+\frac{55}{4}s^4\right)e^2+\frac{1}{8}e^4s^4\Big],\\
A_1 &= \varepsilon^2(2-3s^2)+\varepsilon^4\left[\left(1-11s^2+\frac{19}{2}s^4\right)-\right.\\
&\qquad\left.-e^2\left(5s^2-\frac{13}{2}s^4\right)\right],\\
B_0 &= -\frac{1}{2}\varepsilon^2s^2+\varepsilon^4\left[\left(\frac{3}{2}-6s^2+\frac{37}{8}s^4\right)-e^2\left(s^2-\frac{31}{16}s^4\right)\right],\\
B_1 &= -\frac{\varepsilon^4}{4}(4-5s^2)es^2,\\
B_2 &= -\frac{3\varepsilon^4}{32}e^2s^4.
\end{aligned}\right\} \qquad (4.5)$$

Substituting the values of integrals in (4.1) which we have found, we produce /225
the time equation

$$\bar{n}(t-\bar{t}_0) = 2\arctan\left(\sqrt{\frac{1-\bar{e}}{1+\bar{e}}}\operatorname{tg}\frac{v}{2}\right)-$$
$$-e^*\sqrt{1-\bar{e}^2}\,\frac{\sin v}{1+\bar{e}\cos v}+\gamma_{0,0}v+\gamma_{0,1}\sin v+\gamma_{0,2}\sin 2v+ \qquad (4.6)$$
$$+\bar{\gamma}_{0,0}\varphi+\bar{\gamma}_{2,0}\sin 2\varphi+\bar{\gamma}_{4,0}\sin 4\varphi,$$

in which

$$\bar{n} = \sqrt{\frac{fM}{a^3}}\left\{1-\frac{3}{2}\varepsilon^2(1-e^2)(1-s^2)+\right.$$
$$\left.+\frac{3}{8}\varepsilon^4(1-e^2)(1-s^2)[(1+11s^2)-(1-5s^2)e^2]\right\}, \qquad (4.7)$$
$$e^* = e\{1-\varepsilon^2(1-e^2)(1-s^2)+\varepsilon^4(1-e^2)(1-s^2)(3+e^2)s^2\}, \qquad (4.8)$$

$$
\left.
\begin{aligned}
\gamma_{0,0} &= -(1-e^2)^{3/2}\left\{\frac{1}{2}\varepsilon^2 s^2 - \right. \\
&\qquad \left. - \frac{\varepsilon^4}{16}\left[(24-96s^2+78s^4) - (8-11s^2)s^2e^2\right]\right\}, \\
\gamma_{0,1} &= -\frac{\varepsilon^4}{4}(1-e^2)^{3/2}(4-5s^2)s^2e, \\
\gamma_{0,2} &= \frac{3}{32}\varepsilon^4(1-e^2)^{3/2}s^4e^2, \\
\bar{\gamma}_{0,0} &= s^2(1-e^2)^{3/2}\left\{\frac{1}{2}\varepsilon^2 - \frac{\varepsilon^4}{16}\left[(24-27s^2) - (8-11s^2)e^2\right]\right\}, \\
\bar{\gamma}_{2,0} &= -s^2(1-e^2)^{3/2}\left\{\frac{1}{4}\varepsilon^2 - \frac{\varepsilon^4}{8}\left[(6-7s^2) - (2-3s^2)e^2\right]\right\} \\
\bar{\gamma}_{4,0} &= \frac{\varepsilon^4 s^4}{64}(1-e^2)^{5/2}.
\end{aligned}
\right\} \qquad (4.9)
$$

The time equation can also be represented in the form

$$
\begin{aligned}
\bar{n}(t-\bar{t}_0) &= 2\arctan\left(\sqrt{\frac{1-\bar{e}}{1+\bar{e}}}\tan\frac{v}{2}\right) - \frac{e^*\sqrt{1-\bar{e}^2}\sin v}{1+\bar{e}\cos v} - \\
&\quad - \lambda v + \lambda_{0,1}\sin v + \lambda_{0,2}\sin 2v + \lambda_{2,0}\sin 2u + \\
&+ \lambda_{4,0}\sin 4u + \lambda_{2,-2}\sin(2u-2v) + \lambda_{2,2}\sin(2u+2v),
\end{aligned} \qquad (4.10)
$$

where

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$$
\left.
\begin{aligned}
\lambda &= -\frac{\varepsilon^4}{16}(1-e^2)^{3/2}(24-96s^2+75s^4), \\
\lambda_{0,1} &= -\frac{\varepsilon^4}{4}(1-e^2)^{3/2}(4-5s^2)es^2, \\
\lambda_{0,2} &= \frac{\varepsilon^4}{32}(1-e^2)^{3/2}s^4e^2, \\
\lambda_{2,0} &= -\frac{\varepsilon^2}{4}s^2(1-e^2)^{3/2} + \\
&+ \frac{\varepsilon^4}{16}s^2\left[(12-13s^2) - e^2(4-5s^2)\right](1-e^2)^{3/2}, \\
\lambda_{2,-2} &= -\frac{\varepsilon^4}{32}s^4e^2(1-e^2)^{3/2}, \\
\lambda_{2,2} &= \frac{\varepsilon^4}{32}s^4e^2(1-e^2)^{3/2}, \\
\lambda_{4,0} &= -\frac{\varepsilon^4}{64}s^4(1-e^2)^{5/2}.
\end{aligned}
\right\} \qquad (4.11)
$$

Where $\varepsilon = 0$, this form of the equation is altered to the ordinary equation for the center of the two-body problem.

The methods of solution of equations (4.6) and (4.10) may vary. For

example iteration methods are convenient. Let us represent equation (4.6) in the form

$$f_0(v) + \varepsilon^2 f_2(v) + \varepsilon^4 f_4(v) = \bar{M}, \tag{4.12}$$

where

$$\bar{M} = \bar{n}(t - \bar{t}_0),$$

$$f_0(v) = 2\arctan\left(\sqrt{\frac{1-\bar{e}}{1+\bar{e}}}\operatorname{tg}\frac{v}{2}\right) - \\ - e^*\sqrt{1-\bar{e}^2}\,\frac{\sin v}{1+\bar{e}\cos v} - \lambda v, \tag{4.13}$$

$$f_2(v) = \bar{\lambda}_{2,0}\sin 2u, \tag{4.14}$$

while $f_4(v)$ represents the remaining terms from (4.6). The solution of this equation will be constructed in the form of series with respect to powers of ε. Assuming in this equation the explicitly included ε equal to zero, we come to the equation /227

$$2\arctan\left(\sqrt{\frac{1-\bar{e}}{1+\bar{e}}}\tan\frac{v_0}{2}\right) - e^*\sqrt{1-\bar{e}^2}\,\frac{\sin v_0}{1+\bar{e}\cos v_0} - \lambda v_0 = \bar{M}. \tag{4.15}$$

If we now introduce the supplementary variable (like the eccentric anomaly in the two-body problem)

$$\tan\frac{v_0}{2} = \sqrt{\frac{1+\bar{e}}{1-\bar{e}}}\tan\frac{E}{2}, \tag{4.16}$$

then in place of (4.15) we produce

$$E - e^*\sin E - 2\lambda\arctan\left(\sqrt{\frac{1+\bar{e}}{1-\bar{e}}}\tan\frac{E}{2}\right) = \bar{M}. \tag{4.17}$$

This equation, where $\varepsilon = 0$, corresponds with the Keppler equation. It can be solved by iteration. The convergence of the process of successive approximations is easily established using the principle of compressed images. If we introduce the symbol

$$\Phi(E) = e^*\sin E + 2\lambda\arctan\left(\sqrt{\frac{1+\bar{e}}{1-\bar{e}}}\tan\frac{E}{2}\right), \tag{4.18}$$

then in order to prove the convergence it is sufficient to show that

$$|\Phi'_E(E)|<1.$$

It follows from (4.18) that

$$\Phi'_E(E)=e^*\cos E+\lambda\sqrt{\frac{1+\bar{e}}{1-\bar{e}}}\frac{1-\bar{e}}{1-\bar{e}\cos E}.$$

Keeping in mind that $\bar{e}\geq 0$, we find

$$|\Phi'_E(E)|\leqslant e^*+|\lambda|\sqrt{\frac{1+\bar{e}}{1-\bar{e}}}=K.$$

But for the earth, $\varepsilon < 1/30$, so that

$$K\leq e.$$

The equation (4.15) can be solved by another method, based on a formula similar to the Lagrange formula, used in the solution of the Keppler equation. /228

The uniqueness of the solution of (4.15) can be easily proven using the well-known theorem of the number of roots of an equation [141]

$$F(z)=0,\quad F(z)+\Phi(z)=0. \tag{4.19}$$

If the functions F(z) and $\Phi(z)$ are analytical in the area bounded by closed conture C, continuous in this area and satisfy the inequality $|\Phi(z)| < |F(z)|$ in C, equations (4.19) have identical numbers of roots in this area.

In our problem, assuming

$$F(E)=E-e^*\sin E-\bar{M},$$
$$\Phi(z)=-2\lambda\arctan\left(\sqrt{\frac{1+\bar{e}}{1-\bar{e}}}\tan\frac{E}{2}\right)$$

and keeping in mind the smallness of λ, we discover the equation (4.19) has a single solution. This solution can be represented in the form of a converging series with respect to powers of e* and λ.

First let us analyze the equation

$$z - a - \alpha F(z) - \beta\Phi(z) = 0, \tag{4.20}$$

in which the functions F(z) and $\Phi(z)$ are analytic in a certain area of point z = a. Let us analyze the area limited by circle C with its center at point a of radius r, such that

$$|\alpha F(z) + \beta\Phi(z)| < r.$$

Using the Cauchy integral, it is not difficult to show correctness of the following formula:

$$\Psi(z_0) = \Psi(a) + \sum_{\substack{m=0 \\ |m|+|n|\neq 0}}^{\infty} \sum_{n=0}^{\infty} \frac{\alpha^m \beta^n}{m!n!} \frac{d^{m+n-1}}{da^{m+n-1}} \times \{\Psi'(a)[F(a)]^m [\Phi(a)]^n\}, \tag{4.21}$$

where $\Psi(z)$ is the analytic function of the root z_0 from equation (4.20). Series (4.21) forms the Lagrange formula used in the solution of Keppler equation.

Making the substitutions /229

$$\Psi(E) = E, \ \alpha = e^*, \ \beta = \lambda,$$

$$F(E) = \sin E, \quad \Phi(E) = 2\arctan\left(\sqrt{\frac{1+\bar{e}}{1-\bar{e}}}\,\mathrm{tg}\,\frac{E}{2}\right)$$

and applying formula (4.21), we find that

$$E = \overline{M} + \sum_{m=0}^{\infty} \frac{e^{*m+1}}{(m+1)!} \frac{d^m}{d\overline{M}^m} (\sin^{m+1}\overline{M}) + 2\lambda \arctan\left(\sqrt{\frac{1+\bar{e}}{1-\bar{e}}}\tan\frac{\overline{M}}{2}\right)$$

Here we limit ourselves to the term in the first power of λ since for artificial earth satellites the parameter λ is on the order of 10^{-6}. The second term in (4.22) corresponds to the solution of the ordinary Keppler equation.

§ 5. Determining the Elements of a Satellite Orbit

Determining the elements of the orbit of an artificial earth satellite on the basis of the initial values of the coordinates and velocities has been discussed in articles by Ye. P. Aksenov, Ye. A. Grebenikov and the present author [87], and also in articles by Ye. I. Timoshkova [142] and Ye. P. Aksenov [143]. V. N. Lavrik [144] gives a solution to the boundary-value

problem, on the basis of the total integral of the generalized problem of two immobile centers, in the form advanced by M. D. Kislik. This particular study is quite interesting in that it is based upon Gauss' method for determining orbits in the two-body problem [145].

We shall assume that at the initial moment $t = t_0$, the values of the coordinates x_0, y_0, z_0, as well as of their derivatives with respect to time $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$, are known. From formulas (1.2) we find that

$$\left.\begin{aligned} \xi^2 &= \frac{1}{2}(r^2 - c^2)\{1 + [1 + 4c^2z^2(r^2 - z^2)^{-2}]^{1/2}\}, \\ \eta &= \frac{z}{\xi}, \quad \tan w = \frac{y}{x}. \end{aligned}\right\} \tag{5.1}$$

These formulas enable us to determine the values of the coordinates ξ, η and w at the initial moment. Differentiating formulas (5.1) with respect to time, /230 we obtain

$$\dot{\xi} = \frac{\xi^2 r\dot{r} + c^2 z\dot{z}}{\xi^2 + c^2\eta^2}, \quad \dot{\eta} = \frac{\xi\dot{z} - z\dot{\xi}}{\xi^2}, \quad \dot{w} = \frac{x\dot{y} - y\dot{x}}{r^2 - z^2}, \tag{5.2}$$

from which we determine the initial values of the velocities $\dot{\xi}_0$, $\dot{\eta}_0$, $\dot{w}_0$.

In place of (1.3) - (1.5) we can now write

$$\left.\begin{aligned} c_1 &= (\dot{\xi}^2 + c^2)(1 - \eta^2)\dot{w}, \\ 2h &= J\left(\frac{\dot{\eta}^2}{1 - \eta^2} + \frac{\dot{\xi}^2}{\xi^2 + c^2}\right) + (\xi^2 + c^2)(1 - \eta^2)\dot{w}^2 - \frac{2fM}{J}, \\ 2c_2 &= (1 - \eta^2)^{-1}[2hc^2\eta^2(1 - \eta^2) - c_1^2 - J^2\dot{\eta}^2] \end{aligned}\right\} \tag{5.3}$$

(where, in order to obtain the derivatives of the coordinates with respect to time t, we made use of (1.6)). Substituting in the right-hand members of the equations of (5.3), the initial values of the coordinates of their derivatives, we find the values of the arbitrary constants c_1, c_2 and h.

We must now proceed to the calculation of the elements a, e, s, which can be done with the help of formulas (1.10) and (1.14). These formulas can also be transformed to the following form:

$$\left.\begin{aligned} a &= -\frac{fM}{2h}[1 - \varepsilon^2(1 - e^2)(1 - s^2) + \\ &\qquad + \varepsilon^4(1 - e^2)(3 + e^2)s^2(1 - s^2)], \\ 1 - e^2 &= \frac{4c_2h}{f^2M^2}\{1 - \varepsilon^2(1 + 3e^2)(1 - s^2) + 2\varepsilon^4[1 + 4e^2 + \\ &\qquad + 3e^4 - (1 + 2e^2 + 5e^4)s^2 - 2e^2(1 - e^2)s^4]\}, \end{aligned}\right\} \tag{5.4}$$

$$1 - s^2 = -\frac{c_1^2}{2c_2}\{1 + \varepsilon^2(1 - e^2)s^2 - \\ - \varepsilon^4[(\ldots - 2e^2 - e^4)s^2 - 4(1 - e^2)s^4]\}.$$

The elements a, e, s can be easily determined by the method of successive approximations. Substituting $\varepsilon = 0$ in (5.4), we find approximate values for a, e, s, which we then adopt as our first approximation. It is clear, then, that the second approximation will be sufficiently accurate for our purposes.

We should now proceed to determining the angular elements ω, Ω, $\bar{t}_0$. It should be noted, first, that these elements becomes the corresponding Keplerian elements when $\varepsilon = 0$. Since the auxiliary variables ϕ and v, as defined by formulas (2.1) and (2.2), when $\varepsilon = 0$ coincide with the argument of latitude and of the true anomaly of Keplerian motion, respectively, then the angular element ω coincides with the argument of the pericenter when $\varepsilon = 0$. The angular element Ω when $\varepsilon = 0$ gives the longitude of the ascending node of a Keplerian orbit, because it represents the planetocentric longitude (in the case of an artificial earth satellite, the right ascension of the satellite) at the moment of passage through the equatorial plane of the planet. From the equation of time we can easily determine the meaning of the constant $\bar{t}_0$. Actually, this constant gives us the moment of passage of the satellite through the pericenter of the orbit, with an accuracy on the order of terms containing ε^2. Finally, the element s when $\varepsilon = 0$ coincides with the sine of the inclination i of the satellite orbit. Consequently, it is possible to substitute $s = \sin i$, and then we can write $\sqrt{1 - s^2} = \cos i$. /231

Having determined a, e, s, we make a preliminary calculation of the auxiliary constant ν with the help of formulas (2.8) or (2.7). Since we already know the coordinates r and z when $t = t_0$, then on the basis of the expansions of (2.26) and (2.27), using the method of successive approximations, it is possible to find the values of the angular variables u and v. As a zero approximation we can take their values for the case of Keplerian motion: i.e., for $\varepsilon = 0$. From formulas (2.29) and (2.30), we see that, when $\varepsilon = 0$,

$$a_{ij} = \begin{cases} 1 & \text{if } i = j = 0, \\ 0 & \text{if } i \neq 0,\ j \neq 0, \end{cases} \qquad b_{1i} = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases} \tag{5.5}$$

Therefore, from (2.26) and (2.27), setting $\varepsilon = 0$, we find that

$$\left.\begin{aligned} r &= \frac{p}{1 + \bar{e}\cos v}, \\ z &= \frac{\bar{p}s\sin u}{1 + \bar{e}\cos u}. \end{aligned}\right\} \tag{5.6}$$

The numerical values of v and u obtained on this basis we shall accept as our zero approximation.

The process of successive approximations in this case will be convergent, which can be demonstrated, as previously, on the basis of the principle of compressed mappings.

Following determination of the qualities u and v on the basis of formula (2.10), we calculate the element ω, and on the basis of formula (2.28) obtain a value for the element Ω. /232

Next, on the basis of formula (4.10), it is possible to determine the element $\bar{t}_0$, having made the preliminary substitution $t = t_0$.

As in the case of the two-body problem, it is possible to substitute for the element $\bar{t}_0$ a new element which is the analogue of the mean anomaly at the time of epoch. Formula (4.12) can be transformed as follows:

$$\bar{M} = \bar{n}(t - t_0) + \bar{n}(t_0 - \bar{t}_0). \tag{5.7}$$

Stipulating

$$\bar{M}_0 = \bar{n}(t_0 - \bar{t}_0), \tag{5.8}$$

instead of (5.7) we then have

$$\bar{M} = \bar{n}(t - t_0) + \bar{M}_0. \tag{5.9}$$

The element $\bar{M}_0$ will be found to be more convenient if our purpose is a further refinement in the analytical theory of the motion of a satellite with allowance for other perturbing factors.

Instead of the quantities u and v, we can also use ϕ and v. In this event, the calculation of the elements is carried out on the basis of formulas (2.6), (2.9), (2.10) and (2.15), which will give us a value for ω. Then, from formula (2.24), we can find the longitude of the ascending node Ω.

Still another means of calculating the elements is indicated in the article by Ye. P. Aksenov [143].

Thus, the problem of determining the elements of an orbit has been completely resolved.

In concluding this discussion of the theory of artificial earth satellite motion, as based on the generalized problem of two immobile centers, we add a few concluding remarks. This solution, as distinct from solutions obtained on the basis of classical methods of the theory of perturbations,

embraces perturbations of all orders, from the second and in part from the fourth zonal harmonics. In the case $\delta \neq 0$, the solution will embrace inequalities of any order arising from the asymmetry of the terrestrial field with respect to the equator. If, in addition, we calculate the perturbations arising from the moon, the sun and the resistance of the atmosphere, then even in perturbations of the first order (when making use of the generalized problem of two centers as the intermediate problem) one automatically obtains inequalities arising from the nonlinear superimposition of various perturbing factors. This represents one of the advantages of the proposed method of creating a theory of the motion of artificial earth satellites. Moreover, as has already been noted, in stead of the secular terms of the classical theory of perturbations, the proposed solution embraces only along-period perturbations, which are precisely the ones which generate secular inequalities in the case of the use of approximation methods. The proposed solution, therefore, has made it possible to overcome the basic difficulties of the theory of perturbation associated with small denominators. /233

CHAPTER VI

PERIODIC AND NEAR-PERIODIC MOTIONS OF ARTIFICIAL CELESTIAL BODIES

§ 1. The Poincaré Method. Criteria of Periodicity /234

This chapter is a brief presentation of the essentials of Poincaré small-parameter method, and of the results of the application of that method in celestial ballistics, in connection with periodic and conditional-periodic solutions of differential equations of motion.

First developed by Poincaré in connection with the classical three-body problem, this method is formulated in detail in his outstanding work *Les methodes nouvelles de la mecanique celeste* [146]. It has found wide application not only in celestial mechanics, but also in solid-state mechanics. The method has become particularly flexible owing to the work of Painlevé [147], who, rejecting the use of any physical quantity in the capacity of the small parameter, proposed instead the use of transformed variables dependent upon the small parameter. Similar untraditional approaches to the small-parameter method are found in the works of E. Hopf [148], O. Perron [149], A. A. Orlov [150].

In some instances the small-parameter method enables one to arrive at near-periodic in addition to periodic solutions. In particular, the Poincaré method makes possible bi-frequency conditional-periodic solutions in the limited circular three-body problem, and in the problem of the motion of a material point in the gravitational field of a rotating "triaxial" planet.

Let us consider a system of differential equations of the type

$$\frac{dx_i}{dt} = X_i(t, x_1, \ldots, x_n, \mu) \quad (i = 1, 2, \ldots, n), \tag{1.1}$$

where μ is the small parameter, and the right-hand members of the equations are holomorphic functions of all the variables x_i, t and the parameter μ within a certain region of their values $|x_i| \leq a$, for sufficiently small absolute values of $|\mu| < p$. /235

We shall assume that either a general or a partial solution of the system (1.1) is known for $\mu = 0$:

$$\frac{dx_i}{dt} = X_i(t, x_1, \ldots, x_n, 0) \quad (i = 1, 2, \ldots, n), \tag{1.2}$$

we shall refer to these equation as "simplified"[1]. The solutions of the system (1.2) will be referred to as "generative" (in celestial mechanics a generative solution determines an intermediate orbit).

Let

$$X_{i0} = X_i\,(t,\ x_1,\ldots,\ x_n,\ 0), \tag{1.3}$$

while the known solution of the simplified system we shall denote in the following manner:

$$x_i = x_{i0}\,(t) \quad (i = 1,\ 2,\ldots,\ n), \tag{1.4}$$

it being assumed that this solution satisfies the initial conditions

$$x_i = x_i^{(0)} \qquad (i = 1,\ 2,\ldots,\ n) \quad \text{if } t = t_0. \tag{1.5}$$

In his investigations Poincaré made use of Cauchy's method of majorant functions[2], which is employed in proving the basic theorem of differential equations regarding the existence and uniqueness of solutions. Poincaré's result can be formulated in the form of the following theorem.

Poincaré's Theorem. If the right-hand members of (1.1) are holomorphic functions of their variables within a certain region $|x_i| \leq a$ in the vicinity of $\mu = 0$, and if the system (1.2) admits of a holomorphic solution along the line L on the complex plane t, then system (1.1) has a solution which can be represented in the following series:

$$x_i = \sum_{k=0}^{\infty} \mu^k x_{ik}\,(t), \tag{1.6}$$

which is expanded in increasing powers of μ, and which is absolutely and uniformly convergent on that same line L for a sufficiently small absolute value of μ. /236

For the chosen value of μ, series (1.6) will converge toward a holomorphic function for all values of t which satisfy the condition

$$t_0 \leqslant t \leqslant T, \tag{1.7}$$

where the quantity T depends upon μ. In the general case, the smaller the

[1]The term "simplified equations" (equations simplifieces) was introduced by P. Painlevé.

[2]Cauchy called his method comparison by the calculation of limits.

absolute value of μ, the greater will be the quality T. The proof of this result, based upon Cauchy's method, may be found in [151, 152]. A different method of proof has been given by E. Picard [153].

It should be noted that on the complex plane t, Poincaré's result considerably enlarges the region of existence of the solution with respect to the circle of convergence as determined by Cauchy's theorem, since here majorizing is possible along line L, which cannot contain singular points outside the indicated circle of convergence.

The method of constructing series (1.6) is as follows. We first expand the right-hand members of equation (1.1) in series in powers of μ:

$$X_i = \sum_{k=0}^{\infty} \mu^k X_{ik} \quad (i = 1, 2, \ldots, n). \tag{1.8}$$

Substituting series (1.6) in equation (1.1) and equating the coefficients which have identical powers of μ in the right-hand and in the left-hand members of the equations, we arrive at the following system of equations for successive determination of the functions x_{ik}.

For k = 0:

$$\frac{dx_{i0}}{dt} = X_{i0}(t, x_1, \ldots, x_n, 0), \tag{1.9}$$

For k = 1:

$$\frac{dx_{i1}}{dt} = \sum_{j=1}^{n} \left(\frac{\partial X_i}{\partial x_j}\right)_0 x_{j1} + \mu X_{i1}(t, x_{10}, \ldots, x_{n0}, 0). \tag{1.10}$$

(Here in subsequently $(\partial X_i/\partial x_j)_0$ will denote the values of the derivatives $\partial X_i/\partial x_j$, taken for $x_i = x_{i0}(t)$ (i = 1, 2,..., n).)

It is not difficult to establish that for any value of k the equations for /237 determining x_{ik} retain the same form as that of equation (1.10):

$$\frac{dx_{ik}}{dt} = \sum_{j=1}^{n} \left(\frac{\partial X_i}{\partial x_j}\right)_0 x_{jk} + \Phi_{ik}(t, x_{i0}, \ldots, x_{i,k-1}) \quad (i = 1, 2, \ldots, n), \tag{1.11}$$

where the functions Φ_{ik} depend only upon x_{ik}, as determined from the preceding systems of equations.

Thus, any one of the approximations (terms of the order μ^k comprise the

k-th approximation) can be determined from a system of linear nonhomogeneous equations, which in the general case contain time in explicit form. Consequently, the problem of constructing the series reduces to the integration of a system of differential equations:

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} \left(\frac{\partial X_i}{\partial x_j}\right)_0 y_j \quad (i = 1, 2, \ldots, n), \tag{1.12}$$

which we shall refer to by the term equations in variations.

Generally speaking, when the coefficients $(\partial X_i/\partial x_j)_0$ depend explicitly upon time, the integration of equations in variations is fairly complicated. However, if we know the general solution of the simplified system (1.2), then the general solution of equations in variations (1.12) can be found quite simply. The following theorem applies in this case.

Theorem. If we know the general solution

$$x_i = x_i\ (t, C_1, \ldots, C_n) \tag{1.13}$$

of the simplified system (1.2), in which C_i represents constants of integration, then the general solution of the equations in variations (1.12) can be found by means of differentiating (1.13) with respect to the quantities C_i; this general solution is expressed in the following form:

$$y_i = \sum_{j=1}^{n} \beta_j \frac{\partial x_i}{\partial C_j} \quad (i = 1, 2, \ldots, n), \tag{1.14}$$

where β_j represents arbitrary constants.

Proof. Substituting in (1.2) the general solution (1.13), we arrive at the following system of identities: /238

$$\frac{dx_i(t, C_1, \ldots, C_n)}{dt} \equiv X_i(t, x_1(t, C), \ldots, x_n(t, C), 0). \tag{1.15}$$

Differentiating both members of (1.15) with respect to the arbitrary constants C_i, we arrive at

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial C_j}\right) \equiv \sum_{k=1}^{n} \left(\frac{\partial X_i}{\partial x_k}\right)_0 \frac{\partial x_k}{\partial C_j} \quad (j = 1, 2, \ldots, n), \tag{1.16}$$

which shows that the functions

$$y_{kj} = \frac{\partial x_k}{\partial C_j} \tag{1.17}$$

are partial solutions of the system of equations in variations. Since (1.14) gives a general solution to system (1.2), then

$$\det \left| \frac{\partial x_k}{\partial C_j} \right| \neq 0 \tag{1.18}$$

and hence functions (1.17) constitute a fundamental system of solutions of equations (1.12). Then the general solution of the equations in variations actually will have the form of (1.14).

Returning to system (1.11), we note that its general solution, following determination of the total integral of the equations in variations, is easily found by means of quadratures by the method of variation of arbitrary constants.

Note. Sometimes it is sufficient to know only a certain family of integral curves of equations (1.12), which depends upon a number of integral constants less than n.

For example, let the right-hand members of equation (1.2) be independent of time:

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n, \mu), \tag{1.19}$$

and let

$$x_i = \varphi_i(t) \quad (i = 1, 2, \ldots, n) \tag{1.20}$$

represent the partial solution of the system (1.19) for $\mu = 0$. Then, /23

$$x_i = \varphi_i(t + C) \tag{1.21}$$

also will be solution of the system (1.19) for $\mu = 0$. But from the proof of the preceding theorum, it is clear that

$$y_i = \frac{\partial \varphi_i}{\partial C} \quad (i = 1, 2, \ldots, n) \tag{1.22}$$

will be a solution of the corresponding system of equations in variations. Now instead of (1.22) we can write

$$y_i = \dot{\phi}_i. \tag{1.23}$$

If n = 2, then we can employ (1.23) to find the general solution of equations in variations. Finally, for an arbitrary value of n, the system (1.12) breaks down into n independent equations, and the solution of (1.23)

also enables us to integrate the equations in variations.

Poincaré's theorem, which was referred to at the beginning of this section, was used by its originator as the basis for a method of constructing periodic orbits. The essence of this method is as follows.

Let the system (1.1) satisfy the conditions of the Poincaré theorem, and let it be such that the right-hand members of the equations are periodic functions of time with period T. In addition, let this system, for $\mu = 0$, admit of the periodic solution

$$x_i = \varphi_i(t) \quad (i = 1, 2, \ldots, n), \tag{1.24}$$

which corresponds to the initial conditions

$$t = t_0, \quad x_i = \varphi_i(t_0) \tag{1.25}$$

and is called generative.

We shall denote the period of this solution with the symbol T, so that

$$\varphi_i(t + T) \equiv \varphi_i(t). \tag{1.26}$$

There arises the question of whether there exists a periodic solution to the system (1.1) for $\mu \neq 0$, but fairly small -- a solution which reverts to the solution of (1.24), for $\mu = 0$?

Let us consider the solution of the system (1.1)

$$x_i = \varphi_i(t, \mu). \tag{1.27}$$

Here ϕ_i are such that /240

$$\varphi_i(t, 0) \equiv \varphi_i(t). \tag{1.28}$$

The solution of (1.27) corresponds to the initial conditions

$$t = t_0, \; x_i = \varphi_i(t_0, \mu). \tag{1.29}$$

Let us stipulate that

$$\beta_i = \varphi_i(t_0, \mu) - \varphi_i(t_0). \tag{1.30}$$

In line with Poincaré's theorem, the solution of (1.27) can be represented with the help of series

$$x_i = \varphi_i(t, \beta_1, \ldots, \beta_n, \mu) = \varphi_i(t) + \sum_{j=1}^{n} \left(\frac{\partial \varphi_i}{\partial \beta_j}\right)_0 \beta_j + \left(\frac{\partial \varphi_j}{\partial \mu}\right)_0 \mu + \cdots, \qquad (1.31)$$

which will converge absolutely and uniformly for all values of t, subject to the condition $t \leq \tau(\mu)$, while μ must remain relatively small[1]. Obviously, if the solution of (1.27) has the period $T \leq \tau(\mu)$, then series (1.31) will converge for all values of t.

Thus, it is necessary to find initial conditions such that -- or, what is the same thing, values of β_i -- for which the variables x_i at the moment $T \leq \tau(\mu)$ will assume their initial values (1.29). The necessary and sufficient conditions of periodicity of functions (1.27) are defined as follows:

$$x_i(t_0 + T) - x_i(t_0) = 0, \qquad (1.32)$$

or, in correspondence with (1.27) and (1.29),

$$\psi_i = \varphi_i(t_0 + T, \mu) - \varphi_i(t_0, \mu) = 0. \qquad (1.33)$$

From (1.31) we see that the functions ψ_i depends upon the initial deviations β_i and the parameter μ.

If, from the system of equations (1.33), the qualities β_i can be determined as holomorphic functions of μ, which disappear when $\mu = 0$, then this fact alone will demonstrate the existence of periodic solutions of the system (1.1) of the required form. As follows from the theorem on the existence of inexplicit functions, the quantities β_i can be found, provided /241 the Jacobian

$$\left[\frac{D(\psi_1, \psi_2, \ldots, \psi_n)}{D(\beta_1, \beta_2, \ldots, \beta_n)}\right]_0 \neq 0, \qquad (1.34)$$

while the initial deviations β_i will represent simple solutions of the system of equations (1.33).

In order to calculate the Jacobian (1.34) for $\beta_i = \mu = 0$, it is not

[1]The subscript "0" indicates that the values of the qualities under consideration were chosen for $\beta_i = \mu = 0$.

necessary to know the general solution of equations (1.1). Naturally, it is sufficient to know the linear (with respect to β_i and μ) terms of the expansions of the variables x_i in series.

Note 1. In problems of dynamics we quite frequently come across the case in which the initial system (1.1) admits of first integrals. It is obvious that the existence of first integrals means that the Jacobian (1.34) will be identically equal to zero. Actually, we shall assume that the equations (1.1) possess the first integral

$$\Phi(t, x_1, \ldots, x_n, \mu) = C, \tag{1.35}$$

which is periodic with respect to t (with period T). Then, by reason of (1.27),

$$\Phi(t_0 + T, \varphi_1(t_0 + T, \mu), \ldots, \varphi_n(t_0 + T, \mu), \mu) \equiv \equiv \Phi(t_0, \varphi_1(t_0, \mu), \ldots, \varphi_n(t_0, \mu), \mu) = C,$$

or

$$\Phi(t_0 + T, \varphi_1(t_0 + T, \mu), \ldots, \varphi_n(t_0 + T, \mu), \mu) - - \Phi(t_0, \varphi_1(t_0, \mu), \ldots, \varphi_n(t_0, \mu), \mu) \equiv 0. \tag{1.36}$$

Finally, making allowance for (1.33), we have

$$\Phi(t_0 + T, \varphi_1(t_0, \mu) + \psi_1, \ldots, \varphi_n(t_0, \mu) + \psi_n, \mu) - - \Phi(t_0, \varphi_1(t_0, \mu), \ldots, \varphi_n(t_0, \mu), \mu) \equiv 0. \tag{1.37}$$

Having expanded the left-hand member of (1.37) in series in powers of ψ_i, we find that for $\psi_i = 0$ the left-hand member of (1.37) becomes zero. Consequently, equations (1.33) in the presence of a first integral (1.35) will be independent, so that we can exclude one of the equations, for example $\psi_n = 0$, from consideration. Then the solution of the problem of the existence of periodic solutions reduces to proving the existence of a simple solution of the following system of equations:

$$\psi_1 = \psi_2 = \ldots = \psi_{n-1} = 0 \tag{1.38}$$

with respect to $\beta_1, \beta_2, \ldots, \beta_{n-1}$. For this it is sufficient to require that /242 the Jacobian

$$\left[\frac{D(\psi_1, \psi_2, \ldots, \psi_{n-1})}{D(\beta_1, \beta_2, \ldots, \beta_{n-1})}\right]_0 \tag{1.39}$$

be different from zero. The value of one of the qualities β_i can be chosen on an arbitrary basis.

Note 2. If the system (1.1) is autonomous -- i.e., if

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n, \mu), \tag{1.40}$$

then it is possible to attempt a periodic solution with arbitrary period. Moreover, the periods of the generating solution and of the solution of system (1.40) are by no means necessarily equal. This is indicated by the following considerations.

Let us transform the system (1.40) to a new, independent variable τ:

$$\tau = \frac{t}{T + \alpha(\mu)}, \tag{1.41}$$

where $\alpha(\mu)$ represents an arbitrary holomorphic function which can be represented in the following form:

$$\alpha(\mu) = T \sum_{k=1}^{n} \alpha_k \mu^k \tag{1.42}$$

(here the coefficients α_k remain temporarily undefined).

Then the system (1.40) is transformed to the following form

$$\frac{dx_i}{d\tau} = X_i \cdot (T + \alpha(\mu)), \tag{1.43}$$

and the problem reduces to determining a periodic solution of the system (1.43).

From what has been stated above we see that in proving the existence of periodic functions, the basic role is played by a study of the Jacobian. For dynamic systems, this Jacobian, in the great majority of cases is identically equal to zero. It is natural, therefore, to make use of the specific properties of concrete systems under investigation. Poincaré himself, and also the majority of his successors, under such circumstances made use of the integrals of exact differential equations. However, the number of integrals /243
in some cases turns out to be too small, and the questions of the existence of periodic solutions then either calls for the study of second or even higher

approximations, or else remains entirely obscure.

Now let us consider some other techniques which may be used to simplify the study. For example, it is not difficult to establish whether a given system of differential equations admits of symmetrical solutions. The fact of the existence of symmetrical orbits was made use of by G. Hill [154] in constructing an intermediate lunar orbit (variation curve). This property has also been very widely used by F. R. Moulton [155] in constructing periodic orbits by Poincaré's method.

The symmetry theorem. Let us assume that we have given a system of differential equations

$$\ddot{q}_i = F_i(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) \quad (i = 1, 2, \ldots, n), \tag{1.44}$$

the right-hand members of which are such that the system (1.44) is invariant with respect to the substitution

$$q_i \to q_i, \ \dot{q}_i \to -\dot{q}_i, \ t \to -t \quad (i = 1, 2, \ldots, s), \tag{1.45}$$

$$q_i \to -q_i, \ \dot{q}_i \to \dot{q}_i \quad (i = s+1, \ldots, n). \tag{1.46}$$

Then the conditions of periodicity of the solution will be written as follows:

$$\dot{q}_i(0) = 0, \quad \dot{q}_i\left(\frac{T}{2}\right) = 0 \quad (i = 1, 2, \ldots, s), \tag{1.47}$$

$$q_i(0) = 0, \quad q_i\left(\frac{T}{2}\right) = 0 \quad (i = s+1, \ldots, n), \tag{1.48}$$

where T is the period of the solution.

Proof. Let

$$q_i = q_i(t) \tag{1.49}$$

be the solution of the system (1.44). This solution will be periodic, provided the following conditions are fulfilled:

$$q_i\left(\frac{T}{2}\right) = q_i\left(-\frac{T}{2}\right) \quad (i = 1, 2, \ldots, n). \tag{1.50}$$

Without loss of generality, we assume the moment t = 0 as the initial one.

Let the solution of (1.49) correspond to the following initial conditions: /244

$$q_i(0) = q_i^{(0)}, \quad \dot{q}_i(0) = 0 \qquad (i = 1, 2, \ldots, s), \tag{1.51}$$

$$q_i(0) = 0, \quad \dot{q}_i(0) = \dot{q}_i^{(0)} \qquad (i = s+1, \ldots, n). \tag{1.52}$$

According to the condition adopted in the theorem, equations (1.44) are invariant with respect to the substitution (1.45) - (1.46). Consequently, the solution of (1.49) does not alter when the substitution is made. Therefore, from (1.45) - (1.46) it follows that

$$q_i\left(\frac{T}{2}\right) = q_i\left(-\frac{T}{2}\right), \quad \dot{q}_i\left(\frac{T}{2}\right) = -\dot{q}_i\left(-\frac{T}{2}\right) \tag{1.53}$$
$$(i = 1, 2, , \ldots, s),$$

$$q_i\left(\frac{T}{2}\right) = -q_i\left(-\frac{T}{2}\right), \quad \dot{q}_i\left(\frac{T}{2}\right) = \dot{q}_i\left(-\frac{T}{2}\right) \tag{1.54}$$
$$(i = s+1, \ldots, n).$$

Comparing (1.50) with (1.53) and (1.54), we find that the conditions of periodicity are transformed to the form of (1.47) - (1.48).

§ 2. The Application of Poincaré's Method to Quasi-Liouville Dynamic Systems

Let us consider, first of all, certain peculiarities of the application of Poincaré's small-parameter method to those problems of celestial mechanics and celestial ballistics for which the simplified system of equations of motion satisfies the conditions of the Liouville theorem (see § 5, Chapter 1). Dynamic systems will be referred to with the term "Liouville", provided their general solution in finite form can be determined with the help of Liouville's theorem. We should note that this theorem is applicable to the two-body problem, and also to the classical and to the generalized problem of two immobile centers.

A dynamic system with n degrees of freedom will be referred to with the term quasi-Liouville, provided it is defined by the following formulas for kinetic energy T and force function U:

$$T = \frac{1}{2} A \sum_{i=1}^{n} B_i(Q_i)\, \dot{Q}_i^2, \tag{2.1}$$

$$U = \frac{1}{A} \sum_{i=1}^{n} C_i(Q_i) + \mu W, \tag{2.2}$$

where /245

$$A = \sum_{i=1}^{n} A_i(Q_i). \tag{2.3}$$

In (2.2) μ is a numerically sufficient small parameter, W is a holomorphic function of the generalized coordinates and of the small parameter μ within a

certain region of their variation. The qualities A_i, B_i, C_i are also holomorphic functions of the corresponding variables.

For $\mu = 0$, Liouville's theorem enables us to obtain a general solution of the simplified system of equations. The addition of the perturbing function μW to the potential, in the absence of any special assumptions regarding the structure of the function W, will in the general case prevent obtaining a solution in finite form. The direct application of the Poincaré method leads to complex calculations, since by reason of the conservativeness of the system, the Jacobian (1.33) is identically equal to zero. In addition, the solution of the system of equations in variations, with the help of the second theorem of § 1, is also complex. However, if before hand we perform a number of transformations of the equations of motion, then the construction of solutions by the Poincaré method is considerably simplified. Below is given a method proposed in [156].

Let us proceed now from the generalized coordinates Q_i to the new generalized coordinates q_i with the help of the following relationships:

$$q_i = \int \sqrt{B_i(Q_i)}\, dQ_i \quad (i = 1, 2, \ldots, n). \tag{2.4}$$

Then, in place of (2.1) and (2.2), we will have

$$T = \frac{a}{2} \sum_{i=1}^{n} \dot{q}_i^2, \tag{2.5}$$

$$U = \frac{1}{a} \sum_{i=1}^{n} c_i(q_i) + \mu\omega(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, \mu), \tag{2.6}$$

where a is defined by the formula

$$a = \sum_{i=1}^{n} a_i(q_i). \tag{2.7}$$

The Lagrangian functions of the second type are transformed to the following form /246

$$\frac{d}{dt}(a\dot{q}_i) - \frac{1}{2}\frac{da_i}{dq_i}\sum_{i=1}^{n}\dot{q}_i^2 = \frac{\partial U}{\partial q_i} \quad (i = 1, 2, \ldots, n). \tag{2.8}$$

These admit of a kinetic energy integral:

$$\sum_{i=1}^{n} \dot{q}_i^2 = \frac{2}{a}(U + h). \tag{2.9}$$

. . .

Let us replace the sum of the squares of the generalized velocities in (2.8) with its expression from (2.9), and then instead of (2.8) we arrive at the following

$$\frac{d}{dt}(a\dot{q}_i) = \frac{1}{a}\frac{da_i}{dq_i}(U+h) + \frac{\partial U}{\partial q_i}$$

$$(i = 1, 2, \ldots, n). \tag{2.10}$$

Introducing the new independent variable τ with the help of the relationship

$$a\,\tau = t, \tag{2.11}$$

we reduce (2.10) to the following system of equations:

$$q_i'' = \frac{\partial}{\partial q_i}[a(U+h)] \quad (i = 1, 2, \ldots, n), \tag{2.12}$$

where the prime symbols denote differentiation with respect to τ. From this we obtain the following in explicit form

$$q_i'' = \frac{dc_i}{dq_i} + h\frac{da_i}{dq_i} + \mu\frac{\partial}{\partial q_i}(aW) \quad (i = 1, 2, \ldots, n). \tag{2.13}$$

With $\mu = 0$, equations (2.13) assume the following form:

$$q_i'' = \frac{dc_i}{dq_i} + h\frac{da_i}{dq_i} \quad (i = 1, 2, \ldots, n) \tag{2.14}$$

and can be integrated in quadratures.

Let

$$q_i = q_{i0}(\tau) \quad (i = 1, 2, \ldots, n) \tag{2.15}$$

be a certain partial solution of the system (2.14). Assuming that this solution is generating, we can then look for a solution of the system (2.13) with use of Poincaré's small-parameter method. In this case solution will be in the form of series: /247

$$q_i = q_{i0}(\tau) + \sum_{k=1}^{\infty} \mu^k q_{ik}(\tau). \tag{2.16}$$

In order to determine the functions q_{ik} we can make use of the following differential equations:

$$q''_{ik} - \left(\frac{d^2 c_i}{dq_i^2} + h \frac{d^2 a_i}{dq_i^2} \right)_0 q_{ik} = f_{ik} \begin{pmatrix} i = 1, 2, \ldots, n; \\ k = 1, 2, \ldots \end{pmatrix}, \tag{2.17}$$

where the functions f_{ik} will depend upon q_{i0}, $q_{i1}, \ldots, q_{i,k-1}$. The index "0" means that in every case instead of q_i it is necessary to substitute q_{i0}. Let us transform the equations (2.17) to the new variables z_i:

$$q_{ik} = q'_{i0} z_i \quad (i = 1, 2, \ldots, n). \tag{2.18}$$

Then, instead of (2.17), we will have

$$z''_i q'_{i0} + 2 z'_i q'_{i0} = f_{ik}. \tag{2.19}$$

Multiplying (2.19) by q'_{i0}, and integrating, we arrive at

$$q'^2_{i0} z'_i = \int f_{ik} q'_{i0} d\tau + \alpha_{ik}, \tag{2.20}$$

where the symbol α_{ik} denotes arbitrary constants.

From (2.20) we find the following for z_i:

$$z_i = \int \left[\int f_{ik} q'_{i0} d\tau + \alpha_{ik} \right] \frac{d\tau}{q'^2_{i0}} + \beta_{ik}, \tag{2.21}$$

where β_{ik} represents constants of integration.

From (2.18) and (2.21), we know that

$$q_{ik} = q'_{i0} \left\{ \int \frac{1}{q'^2_{i0}} \left[\int f_{ik} q'_{i0} d\tau + \alpha_{ik} \right] d\tau + \beta_{ik} \right\}. \tag{2.22}$$

Thus, any of the functions q_{ik} can be found by means of quadratures.

§ 3. Periodic Motions of a Satellite in the Gravitational Field of a Slowly Rotating Planet

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Let us assume that the satellite, considered as a material point, is moving in the gravitational field of a planet which is slowly rotating around one of its major axes of inertia at constant velocity. We shall assume, further, that

the planet possesses dynamic symmetry with respect to the plane passing through the center of mass and lying perpendicular to the planet's axis of rotation.

We shall adopt a rectangular system of coordinates with origin at the planet's center of inertia, the coordinate axes being directed along the main axes of inertia. Then the gravitational potential of the planet at an external point will be expressed with the help of the following formula (see § 3, Chapter 2):

$$U = \frac{fM}{r}\left\{1 + \sum_{k=2}^{\infty}\left(\frac{d}{r}\right)^k Q_k(x, y, z)\right\}, \tag{3.1}$$

where M is the mass of the planet, d is the radius-vector of the most distant point, $r = \sqrt{x^2 + y^2 + z^2}$, and $Q_k(x, y, z)$ represents similar harmonic polynomials of the k-th degree with respect to x, y, z. These polynomials are easily obtained by transforming formulas (3.15) Chapter 2 from spherical coordinates to rectangular coordinates. Thus, for example, on the basis of (3.20) Chapter 2, for $Q_2(x, y, z)$ we obtain

$$Q_2 = \frac{1}{2M}\{(C + B - 2A)x^2 + (A + C - 2B)y^2 + (B + A - 2C)z^2\}. \tag{3.2}$$

Let us consider the specific case in which the planet is rotating around the z-axis with angular velocity n. Then, according to (1.24) Chapter 1, the equations of motion will be as follows:

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} - n^2x &= \frac{\partial U}{\partial x}, \\ \ddot{y} + 2n\dot{x} - n^2y &= \frac{\partial U}{\partial y}, \\ \ddot{z} &= \frac{\partial U}{\partial z}. \end{aligned}\right\} \tag{3.3}$$

Since the planet is dynamically symmetrical with respect to the plane Oxy, then /249 the potential U will be an even function of coordinate z, and therefore the equations of motion (3.3) will admit of plane motions of the satellite in the equatorial plane z = 0. Let us consider these motions in the light of [157].

Let us transfer, now, to the dimensionless variables ξ and η, using the formulas

$$x = b\xi, \quad y = b\eta,$$

in which b is an undefined constant quantity. The equation for these dimensionless parameters are written as follows:

$$\left.\begin{aligned} \ddot{\xi} - 2n\dot{\eta} - n^2\xi &= \frac{\partial V}{\partial \xi}, \\ \ddot{\eta} + 2n\dot{\xi} - n^2\eta &= \frac{\partial V}{\partial \eta}, \end{aligned}\right\} \tag{3.4}$$

while for the force function we have

$$V = \frac{k^2}{\rho}\left[1 + \sum_{n=2}^{\infty}\left(\frac{d}{b}\right)^n \frac{\overline{Q}_n(\xi, \eta)}{\rho^{2n}}\right],$$

where the following notations are adopted

$$k^2 = \frac{fM}{b^3}, \qquad \rho = \sqrt{\xi^2 + \eta^2},$$

while $\overline{Q}_n$ represents polynomials, analogous to Q_n.

Let us introduce into the equations of motion the small parameter μ, assuming that

$$n = \nu\mu, \qquad \left(\frac{d}{b}\right)^n = \gamma\mu^{n-1},$$

where ν and γ represent certain constants. Then the system (3.4) is transformed to the following form:

$$\left.\begin{aligned} \ddot{\xi} - 2\mu\nu\dot{\eta} - \mu^2\nu^2\xi &= \frac{\partial \overline{V}}{\partial \xi}, \\ \ddot{\eta} + 2\mu\nu\dot{\xi} - \mu^2\nu^2\eta &= \frac{\partial \overline{V}}{\partial \eta}, \end{aligned}\right\} \tag{3.5}$$

where

$$\overline{V} = \frac{k^2}{\rho}\left[1 + \gamma\sum_{n=2}^{\infty}\mu^{n-1}\frac{\overline{Q}_n(\xi, \eta)}{\rho^{2n}}\right]. \tag{3.6}$$

We next introduce the elliptical coordinates u, v, which are defined by the following formulas /250

$$\left.\begin{aligned} \xi &= \operatorname{ch} v \cos u - 1, \\ \eta &= -\operatorname{sh} v \sin u, \end{aligned}\right\} \tag{3.7}$$

and also the independent variable τ:

$$dt = (\mathrm{ch}^2 v - \cos^2 u)\ \tau. \tag{3.8}$$

Then, in place of the system (3.5), we arrive at the following:

$$\left.\begin{aligned} u'' &= 2\mu\nu J v' + W_u', \\ v'' &= -2\mu\nu J u' + W_v', \end{aligned}\right\} \tag{3.9}$$

where the force function W is defined as follows:

$$W = k^2(\mathrm{ch}\, v + \cos u) + \frac{h}{2}(\mathrm{ch}\, 2v - \cos 2u) + \frac{\mu^2\nu^2}{2}(\mathrm{ch}\, v - \cos u)^2 + J\overline{W}. \tag{3.10}$$

In formulas (3.9) - (3.10) we have made use of the following designations:

$$J = \mathrm{ch}^2 v - \cos^2 u,\ \rho = \mathrm{ch}\, v - \cos u,$$
$$\overline{W} = k^2 \sum_{n=2}^{\infty} \mu^{n-1} \frac{Q_{2n}(u, v)}{\rho^{2n+1}},$$

where h is the constant of the Jacobi integral which is admitted in system (3.3).

With $\mu = 0$, the system (3.10) reduces to the following form:

$$\left.\begin{aligned} u_0'' &= -k^2 \sin u_0 + h \sin 2u_0, \\ v_0'' &= k^2\, \mathrm{sh}\, v_0 + h\, \mathrm{sh}\, 2v_0. \end{aligned}\right\} \tag{3.11}$$

These equations constitute a simplified system which admits of a family of solutions:

$$\left.\begin{aligned} v_0 &= \mathrm{const}, \\ u_0 &= 2\arctan\left(\mathrm{cth}\, \frac{v_0}{2} \tan\sigma\tau\right), \end{aligned}\right\} \tag{3.12}$$

where v_0 denotes the root of the equation

$$\mathrm{ch}\, v_0 = -\frac{k^2}{2h}, \tag{3.13}$$

while σ is defined by the formula

$$\sigma^2 = \frac{k^2 \operatorname{sh}^2 v_0}{4 \operatorname{ch} v_0}.$$

According to (3.31) Chapter 1, the equation $v_0 = \text{const}$ defines an ellipse. Consequently, the solution of (3.12) represents the motion of the satellite in an elliptical orbit, whose measure semi-axis is equal to $\operatorname{ch} v_0$, and whose eccentricity is equal to $1/\operatorname{ch} v_0$. From (3.11) it is not difficult to determine that:

$$\left.\begin{aligned} \cos u_0 &= \frac{\operatorname{ch} v_0 \cos 2\sigma\tau - 1}{\operatorname{ch} v_0 - \cos 2\sigma\tau}, \\ \sin u_0 &= \frac{\operatorname{sh} v_0 \sin 2\sigma\tau}{\operatorname{ch} v_0 - \cos 2\sigma\tau}, \\ u_0' &= \frac{2\sigma \operatorname{sh} v_0}{\operatorname{ch} v_0 - \cos 2\sigma\tau}. \end{aligned}\right\} \tag{3.14}$$

Now we shall assume that

$$u = u_0 + \bar{u}, \qquad v = v_0 + \bar{v}. \tag{3.15}$$

By means of transforming (3.15), we arrive at differential equations for $\bar{u}$ and $\bar{v}$ (see § 1, this chapter):

$$\left.\begin{aligned} \bar{u}'' &= f(\bar{u}, \bar{v}, \bar{u}', \bar{v}', \tau, \mu), \\ \bar{v}'' &= g(\bar{u}, \bar{v}, \bar{u}', \bar{v}', \tau, \mu), \end{aligned}\right\} \tag{3.16}$$

which, obviously, will satisfy all of the conditions of the Poincaré theorem. The right-hand members of equations (3.16) are periodic with respect to τ with period $T = \pi/\sigma$. Therefore, the Poincaré theorem regarding the construction of solutions in the form of series in powers of the small parameter μ can be applied to the system (3.16).

Thus, we can look for a solution in the form of series:

$$\bar{u} = \sum_{k=1}^{\infty} \mu^k u_k(\tau), \qquad \bar{v} = \sum_{k=1}^{\infty} \mu^k v_k(\tau). \tag{3.17}$$

The variables $\bar{u}$ and $\bar{v}$ must be subject to the conditions of periodicity:

$$\left.\begin{aligned} \bar{v}(T) - \quad (0) = 0, \qquad \bar{u}(T) - u(0) = 0, \\ \bar{v}'(T) - \bar{v}'(0) = 0, \qquad \bar{u}'(T) - \bar{u}'(0) = 0. \end{aligned}\right\} \tag{3.18}$$

If from the system (3.18) it is possible to determine the arbitrary constants of the general solution of equations (3.16) as singled valued functions of the parameter μ, then the series (3.17) will represent a periodic solution of the system (3.10). /252

Since the equations of motion admit of a Jacobi integral, then by reason of Note 1, § 1, equations (3.18) will be dependent. That being true, it is sufficient to consider only three of the equations of (3.18), having chosen one of the constants of integration in arbitrary fashion. According to (1.38)-(1.39) the condition of periodicity reduces to the Jacobian being equal to zero (the Jacobian consists of the right-hand members of three of the equations of (3.18) with respect to three arbitrary constants; it is calculated for zero values of μ and of the arbitrary constants in question).

In order to compile the Jacobian, it is necessary to find the general solution of the equations of the first approximation:

$$\left.\begin{aligned} u_1'' &= -k^2\left(\frac{\cos 2u_0}{\operatorname{ch} v_0}+\cos u_0\right)u_1+\left(\frac{\partial\overline{W}_1}{\partial u}\right)_0, \\ v_1'' &= -k^2\frac{\operatorname{sh}^2 v_0}{\operatorname{ch} v_0}v_1-2\nu\left(\operatorname{ch}^2 v_0-\cos^2 u_0\right)u_0'+\left(\frac{\partial\overline{W}_1}{\partial v}\right)_0. \end{aligned}\right\} \tag{3.19}$$

The general solution of the system (3.19) is given by formulas (2.22), since the simplified system (3.11) satisfies the condition of the Liouville theorem:

$$\left.\begin{aligned} v_1 &= \beta_1\cos\omega\tau+\beta_2\sin\omega\tau+\Phi_1(\tau), \\ u_1 &= u_0'\int\left[u_0'\left(\frac{\partial\overline{W}_1}{\partial u}\right)_0 d\tau+\beta_3\right]\frac{d\tau}{u_0'^2}+u_0'\beta_4, \end{aligned}\right\} \tag{3.20}$$

where β_1, β_2, β_3, β_4 are constants of integration, while

$$\omega^2=\frac{k^2\operatorname{sh}^2 v_0}{\operatorname{ch} v_0}. \tag{3.21}$$

Making allowance for (3.12) - (3.13), we find that

$$u_1=u_0'\frac{1+2\operatorname{ch}^2 v_0}{8\sigma^2\operatorname{sh}^2 v_0}\cdot\beta_3\tau+\Phi_2(\tau). \tag{3.22}$$

In (3.20) and (3.22) it is evident that Φ_1 and Φ_2 are periodic functions of τ with period T.

Now let us write out the conditions of periodicity of (3.18) in explicit form: /253

$$\left.\begin{aligned}
\psi_1 &= \frac{\pi(1+2\operatorname{ch}^2 v_0)}{\sigma^2 \operatorname{sh} v_0 (\operatorname{ch} v_0 - 1)}\beta_3 + \cdots = 0,\\
\psi_2 &= \left(\cos\frac{\pi\omega}{\sigma} - 1\right)\beta_1 + \sin\frac{\pi\omega}{\sigma}\cdot\beta_2 + \cdots = 0,\\
\psi_3 &= -\omega\sin\frac{\pi\omega}{\sigma}\beta_1 + \omega\left(\cos\frac{\pi\omega}{\sigma} - 1\right)\beta_2 + \cdots = 0.
\end{aligned}\right\} \qquad (3.23)$$

The condition $\psi_4 = u_1'(\tau) - u'(0) = 0$, which drops out from (3.23), is fulfilled identically in the first approximation. Therefore, from (3.19) - (3.23), we can deduce the existence of periodic solutions only in the first approximation. If, however, oddness of the potential with respect to y and evenness with respect to x is required, by allowing for the invariance of the equations with respect to the substitution $x(t) \to x(-t)$, $y(t) \to -y(-t)$, $t \to -t$, with the help of (1.53) - (1.54) and the Jacobi integral, we obtain the following new conditions of invariance

$$\psi_i^*(\beta_2, \beta_3, \beta_4) = 0 \qquad (i = 1, 2, 3), \qquad (3.24)$$

for which the Jacobian is equal to

$$\frac{D(\psi_1^*, \psi_2^*, \psi_3^*)}{D(\beta_2, \beta_3, \beta_4)} = 2\pi\sigma\omega. \qquad (3.25)$$

As is evident from (3.25), for any generating elliptical orbits, with the exception of the denumerable set, there exist periodic solutions of the exact equations of motion. Thus, in the case under consideration, on the basis of (3.25) the system (3.9) admits of periodic solutions with the period

$$T = \frac{2\sqrt{\operatorname{ch} v_0}}{k \operatorname{sh} v_0}. \qquad (3.26)$$

These solutions depend upon two arbitrary constants, b and t_0, where t_0 represents the moments of passage of the satellite through the x-axis. The dimensions of the orbits are characterized by the quantities b and v_0 (v_0 also characterizes the form of the orbit). For sufficiently large values of b, the constructed series will converge in correspondence with Poincaré's theorem.

We should note that the motions found with respect to the rotating system of coordinates will be almost periodic, since the relationship between the independent variable τ and the time t is given by the formula /254

$$\tau = \lambda(t - t_0) + F(\tau), \qquad (3.27)$$

where λ is a certain constant, while $F(\tau)$ is a periodic function of τ with a period which, generally speaking, is incommensurable with the quantity $2\pi/\lambda$. Transferring to a system of coordinates with fixed directions of the coordinate axes leads to the appearance of trigonometric terms with period $2\pi/n$ in the

formulas for the coordinates of the satellite. Therefore, in the general case satellite motions will be near-periodic, depending upon three periods.

Theorem. To almost all (in the sense of degree) generating periodic orbits whose elements satisfy the conditions

$$\left.\begin{aligned} &a = b\,\mathrm{ch}\,v_0 > a_0, \quad 0 < e_0 < e = \frac{1}{\mathrm{ch}\,v_0} < 1, \quad i = 0, \\ &\omega = \pi s/2 \ (s = 0, 1, 2, \ldots), \quad -\infty < t_0 < +\infty, \end{aligned}\right\} \tag{3.28}$$

where a is the major semi-axis, e is the eccentricity, ω is the angular distance of the pericenter, i is the inclination of the orbit to the equatorial plane, t_0 is the moment of passage through the pericenter, and a_0, e_0 are certain constants, there correspond a near-periodic orbits with two periods.

Corollary. In the equatorial plane of a planet which possesses dynamic symmetry with respect to the axis of rotation and to the plane which is parallel thereto, passing through the center of inertia, nearly all motions are near-elliptical, with the exception, possibly, of motions taking place rather close to the surface of the planet. Actually, since the equations of motion are invariant with respect to transformation of the rotation around the axis by a common angle, the angular distance of the pericenter is an arbitrary matter.

Note. Since the equations of the limited angular problem of three-bodies have the same form as the equations of (3.3) (see (1.25) - (1.26)), then the results obtained can be extended to apply to orbits of the secular three-body problem.

§ 4. Near-Secular Satellite Orbits

Since it is difficult to construct periodic orbits with due allowance for all of the various perturbing factors which exerts a perceptible effect, the solution of the problem can be broken down into two parts. First we construct a periodic orbit in the simplified problem the equations of which /255
contain all of the basic perturbing factors; this orbit is assumed to be intermediate. Next we determine the inequalities which are caused by the perturbing forces which were not taken into consideration in the simplified problem [158].

Let us consider the motion of an artificial earth satellite coming under the influence of the gravitational forces of the earth, the moon and the sun. We shall solve the problem on the assumption that the moon and sun are moving with respect to the earth along secular orbits the plane of the ecliptic with constant angular velocity. Let m_1, m_2, m_3 represent the masses of the earth the moon and the sun respectively, and let R, a_1, a_2, represent the geocentric

distances of the satellite, the moon and the sun. Then, in a geocentric rectangular system of coordinates with fixed directions of its axes, and whose basic plane is assumed to be the plane of the ecliptic, the equations of motion of the artificial earth satellite will be as follows:

$$\left.\begin{aligned} \ddot{X} &= -\frac{fm}{R^3}X + \frac{\partial}{\partial X}(V_1 + V_2), \\ \ddot{Y} &= -\frac{fm}{R^3}Y + \frac{\partial}{\partial Y}(V_1 + V_2), \\ \ddot{Z} &= -\frac{fm}{R^3}Z + \frac{\partial}{\partial Z}(V_1 + V_2), \end{aligned}\right\} \tag{4.1}$$

where V_1 and V_2 represent perturbing functions which characterize the perturbing influence of the moon and of the sun:

$$V_1 = fm_1\left(\frac{1}{\Delta_1} - \frac{XX_1 + YY_1}{a_1^3}\right), \tag{4.2}$$

$$V_2 = fm_2\left(\frac{1}{\Delta_2} - \frac{XX_2 + YY_2}{a_2^3}\right). \tag{4.3}$$

In the previous two equations, X_1, Y_1 are the coordinates of the moon, and X_2, Y_2 are the coordinates of the sun. If the conjunction of moon and sun is taken to mark the beginning of time calculation, then for the coordinates of the moon and sun we obtain the following:

$$X_1 = a_1 \cos n_1 t, \qquad Y_1 = a_1 \sin n_1 t, \tag{4.4}$$

$$X_2 = a_2 \cos n_2 t, \qquad Y_2 = a_2 \sin n_2 t \tag{4.5}$$

while for the distances Δ_i we will have /256

$$\Delta_i^2 = a_i^2 - 2a_i(X \cos n_i t + Y \sin n_i t) + R^2 \quad (i = 1, 2). \tag{4.6}$$

We now relate the motion of the artificial earth satellite to a new geocentric system of coordinates ξ, η, ζ, which rotates with constant angular velocity n_0 with respect to axis Z in the direction of motion of the earth and sun. For n_0 we can adopt in particular, either n_1 or n_2. In addition, for the unit of length we shall adopt the radius of the unperturbed secular orbit a of the satellite. The formulas for transformation of coordinates are written as follows:

$$\left.\begin{aligned} X &= a(\xi \cos n_0 t - \eta \sin n_0 t), \\ Y &= a(\xi \sin n_0 t + \eta \cos n_0 t), \\ Z &= a\zeta. \end{aligned}\right\} \tag{4.7}$$

The system of equations of motion is transformed to the following form:

$$\left.\begin{aligned} \ddot{\xi} - 2n_0\dot{\eta} - n_0^2\xi &= -\frac{fm\xi}{a^3\rho^3} + \frac{\partial}{\partial\xi}(\bar{V}_1 + \bar{V}_2), \\ \ddot{\eta} + 2n_0\dot{\xi} - n_0^2\eta &= -\frac{fm\eta}{a^3\rho^3} + \frac{\partial}{\partial\eta}(\bar{V}_1 + \bar{V}_2), \\ \ddot{\zeta} &= -\frac{fm\zeta}{a^3\rho^3} + \frac{\partial}{\partial\zeta}(\bar{V}_1 + \bar{V}_2), \end{aligned}\right\} \tag{4.8}$$

where

$$\rho = \frac{R}{a}, \quad \bar{V}_i = \frac{1}{a^2}V_i \quad (i = 1, 2).$$

In this new system of coordinates, instead of (4.6) we will have

$$\Delta_i^2 = a_i^2 - 2aa_i\,[\xi\cos(n_i - n_0)\,t + \eta\sin(n_i - n_0)\,t] + a^2\rho^2. \tag{4.9}$$

Since a < a_i, the perturbing functions $\bar{V}_i$ may be expanded in absolutely and uniformly converging series of Legendre polynomials. Introducing the designations

$$\left.\begin{aligned} \mu\alpha_i &= \frac{a}{a_i}, \\ \cos\varphi_i &= \frac{1}{\rho}[\xi\cos(n_i - n_0)\,t + \eta\sin(n_i - n_0)\,t], \end{aligned}\right\} \tag{4.10}$$

we shall represent Δ_i in the form

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$$\Delta_i^2 = a_i^2[1 - 2\mu\alpha_i\rho\cos\varphi_i + \mu^2\alpha_i^2\rho^2]. \tag{4.11}$$

Taking into considering (4.10) - (4.11) for those quantities which are inverse with respect to Δ_i, we have

$$\frac{1}{\Delta_i} = \frac{1}{a_i}\sum_{s=0}^{\infty}\mu^s\alpha_i^s\rho^s P_s(\cos\varphi_i). \tag{4.12}$$

Then the desired expansions will have the following form:

$$\bar{V}_i = \frac{fm_i a_i}{a_i^2}\sum_{s=1}^{\infty}\mu^s\alpha_i^s\rho^{s+1}P_{s+1}(\cos\varphi_i). \tag{4.13}$$

We then perform still another transformation, introducing the new independent variable

$$\tau = (n - n_0)t, \tag{4.14}$$

where n is the mean unperturbed siderial motion of the satellite. Introducing also the following designations,

$$\left.\begin{aligned} \nu &= \frac{n_0}{n - n_0}, \\ k &= \frac{fm}{a^3 (n - n_0)^2}, \\ k_i &= \frac{fm_i\alpha_i}{a_i^2 (n - n_0)^2}, \end{aligned}\right\} \tag{4.15}$$

we find that the system of equations of motion (4.8) is transformed to the following form

$$\left.\begin{aligned} \xi'' - 2\nu\eta' - \nu^2\xi &= -\frac{k\xi}{\rho^3} + \frac{\partial W}{\partial \xi}, \\ \eta'' + 2\nu\xi' - \nu^2\eta &= -\frac{k\eta}{\rho^3} + \frac{\partial W}{\partial \eta}, \\ \ddot{\zeta} &= -\frac{k\zeta}{\rho^3} + \frac{\partial W}{\partial \zeta}, \end{aligned}\right\} \tag{4.16}$$

where the perturbing function W is defined as follows: /258

$$W = \sum_{s=1}^{\infty} [k_1\alpha_1^s P_{s+1}(\cos\varphi_1) + k_2\alpha_2^s P_{s+1}(\cos\varphi_2)]\,\rho^{s+1}\mu^s. \tag{4.17}$$

Finally, we apply the following transformation:

$$\left.\begin{aligned} \xi &= (1 + x)\cos\tau - y\sin\tau, \\ \eta &= (1 + x)\sin\tau + y\cos\tau, \\ \zeta &= z. \end{aligned}\right\} \tag{4.18}$$

In the new variables, in place of (4.16), we have the following system of equations

$$\left.\begin{aligned} x'' - 2(1+\nu)y' - (1+\nu)^2(1+x) &= -\frac{k(1+x)}{\rho^3} + \frac{\partial W}{\partial x}, \\ y'' + 2(1+\nu)x' - (1+\nu)^2 y &= -\frac{ky}{\rho^3} + \frac{\partial W}{\partial y}, \\ z'' &= -\frac{kz}{\rho^3} + \frac{\partial W}{\partial z}, \end{aligned}\right\} \tag{4.19}$$

while now $\rho^2 = (1 + x)^2 + y^2 + z^2$.

We now convert to the new variables $\cos \phi_i$, which appear in W:

$$\cos\varphi_i = \frac{1}{\rho}\left[(1+x)\cos\frac{n_i - n}{n - n_0}\tau + y\sin\frac{n_i - n}{n - n_0}\tau\right].$$

The resulting system (4.19) has right-hand members which are periodic with respect to τ, provided the quantities $n_1 - n$ and $n_2 - n$ are commensurable; that is, when

$$n = \frac{pn_2 - qn_1}{p - q} \quad (p \neq q), \tag{4.20}$$

where p and q are any mutually prime numbers.

If n is chosen in correspondence with the condition of (4.20), then we are able to apply the Poincaré method in searching for periodic solutions of system (4.19). As the small parameter we shall adopt the quantity μ. The simplified system obtained from (4.19) with $\mu = 0$ is as follows: /259

$$\left.\begin{aligned} x'' - 2(1+\nu)y' - (1+\nu)^2(1+x) &= -\frac{k(1+x)}{\rho^3}, \\ y'' + 2(1+\nu)x' - (1+\nu)^2 y &= -\frac{ky}{\rho^3}, \\ z'' &= -\frac{kz}{\rho^3}. \end{aligned}\right\} \tag{4.21}$$

This system admits of a trivial solution, provided the following condition is fulfilled:

$$n^2 = \frac{fm}{a^3}, \tag{4.22}$$

(this condition is fulfilled with the appropriate choice of a).

In the variables ξ, η, ζ the trivial solution under consideration has the following form:

$$\xi = \cos\tau, \ \eta = \sin\tau, \ \zeta = 0. \tag{4.23}$$

It is evident that to the generating solutions in this case there correspond Keplerian secular orbits of radius a.

Assuming that this solution is generating, we shall look for periodic solutions of the system (4.19) with period T, equal to the period of the right-hand members of equations. Examining the ordinary conditions of periodicity of (1.33) does not resolve the problem of the existence of periodic solutions, since the corresponding Jacobian becomes zero, while system (4.19) does not admit of any integrals. However, the system of equations (4.19) is invariant with respect to the substitution

$$x \to x,\ y \to -y,\ z \to z,\ \tau \to -\tau, \tag{4.24}$$

and therefore by reason of the symmetry theorem (see (1.44) - (1.48)) the conditions of periodicity reduce to the following form:

$$\left.\begin{aligned} &\psi_1 = x'\left(\frac{T}{2}\right) = 0,\ \psi_3 = y\left(\frac{T}{2}\right) = 0,\ \psi_5 = z'\left(\frac{T}{2}\right) = 0,\\ &\psi_2 = x'(0) = 0,\quad \psi_4 = y(0) = 0,\quad \psi_6 = z'(0) = 0. \end{aligned}\right\} \tag{4.25}$$

Let us consider the equations of the first approximation: /260

$$\left.\begin{aligned} &x'' - 2(1+\nu)y' - 3(1+\nu)^2 x = \\ &\qquad = b_1 + c_1 \cos 2\frac{n_1 - n}{n - n_0}\tau + d_1 \cos\frac{n_2 - n}{n - n_0}\tau,\\ &y'' + 2(1+\nu)x' = c_2 \sin 2\frac{n_1 - n}{n - n_0}\tau + d_2 \sin 2\frac{n_2 - n}{n - n_0}\tau,\\ &z'' + (1+\nu)^2 z = 0. \end{aligned}\right\} \tag{4.26}$$

The general solution of this system will be

$$\left.\begin{aligned} x &= \frac{b_1 + \beta_1}{(1+\nu)^2} + \beta_2 \cos(1+\nu)\tau + \beta_3 \sin(1+\nu)\tau + \\ &\quad + \bar{c}_1 \cos 2\frac{n_1 - n}{n - n_0}\tau + \bar{d}_1 \cos 2\frac{n_2 - n}{n - n_0}\tau,\\ y &= -2\frac{b_1 + \beta_1}{1+\nu}\tau - 2\beta_2 \sin(1+\nu)\tau + 2\beta_3 \cos(1+\nu)\tau + \\ &\quad + \beta_4 + \bar{c}_2 \sin 2\frac{n_1 - n}{n - n_0}\tau + \bar{d}_2 \sin 2\frac{n_2 - n}{n - n_0}\tau,\\ z &= \beta_5 \cos(1+\nu)\tau + \beta_6 \sin(1+\nu)\tau, \end{aligned}\right\} \tag{4.27}$$

where β_i represents the constants of integration.

The coefficients of equations (4.26) and (4.27) can be easily expressed in terms of known quantities. From (4.25) - (4.27) we obtain the conditions of periodicity:

$$\left.\begin{aligned}
&\psi_1 = -\beta_2(1+\nu)\sin\frac{1+\nu}{2}T+\beta_3(1+\nu)\cos\frac{1+\nu}{2}T- \\
&-2\bar{c}_1\frac{n_1-n}{n-n_0}\sin\frac{n_1-n}{n-n_0}T-2\bar{d}_1\frac{n_2-n}{n-n_0}\sin\frac{n_2-n}{n-n_0}T+ \\
&\qquad\qquad +\cdots=0, \\
&\psi_2 = (1+\nu)\beta_3+\cdots=0, \\
&\psi_3 = -\frac{b_1+\beta_1}{1+\nu}T-2\beta_2\sin\frac{1+\nu}{2}T+2\beta_3\cos\frac{1+\nu}{2}T+ \\
&\quad +\beta_4+\bar{c}_2\sin\frac{n_1-n}{n-n_0}T+\bar{d}_2\sin\frac{n_2-n}{n-n_0}T+\cdots=0, \\
&\psi_4 = 2\beta_3+\beta_4+\cdots=0, \\
&\psi_5 = -\beta_5(1+\nu)\sin\frac{1+\nu}{2}T+\beta_6(1+\nu)\cos\frac{1+\nu}{2}T+ \\
&\qquad\qquad +\cdots=0, \\
&\psi_6 = \beta_6(1+\nu)+\cdots=0.
\end{aligned}\right\} \qquad (4.28)$$

The Jacobian of these equations with respect to the arbitrary constants β_i is equal to /261

$$\left[\frac{D(\psi_1,\psi_2,\ldots,\psi_6)}{D(\beta_1,\beta_2,\ldots,\beta_6)}\right]_0 = -(1+\nu)^3 T\sin^2\frac{1+\nu}{2}T. \qquad (4.29)$$

This Jacobian, obviously, is different from zero, provided that

$$(1+\nu)\,T \neq 2\pi s,$$

where s is any whole number.

From this we conclude that with the appropriate choice of initial conditions it is possible to construct a satellite orbit which will be periodic within the coordinate system which we have chosen. The series which represent the periodic solution will be convergent for a sufficiently small absolute value of μ: i.e., for generating orbits, whose radii are sufficiently small in comparison with the distance from the earth to the moon and to the sun.

Note. The foregoing reasoning is also justified in the case in which additional perturbing forces resulting from the particular shape of the earth are taken into consideration, on the assumption that the figure of the earth is a spheroid.

§ 5. Periodic Orbits of an Artificial Satellite of the Moon

We shall now consider the motion of an artificial satellite of the moon moving under the gravitational attraction of the moon and the earth. Here, as was done previously in the work by Ye. P. Aksenov and V. G. Demin [159], we shall make the following assumptions.

1. The centers of mass of the earth and moon are moving in circular Keplerian orbits around a common center of inertia at an angular velocity n', defined by the formula

$$n'^2 = \frac{f(M + M_1)}{c^3}. \tag{5.1}$$

2. The artificial lunar satellite is a passively gravitating material point.

3. The earth is a sphere with spherical distribution of density which attracts according to Newtonian law.

4. The moon is a rigid body which rotates around an axis which is perpendicular to the plane of the lunar orbit, and which possesses symmetry with respect to the plane of the lunar orbit[1]. /262

We shall now introduce a selenocentric system of rectangular coordinates, the basic plane of which is assumed to be the plane of the lunar orbit. To this coordinate system we shall communicate the uniform rotation around the z-axis with angular velocity of n' in the direction of the earth's rotation; while the x-axis will be made to pass through the center of the earth. The equation of motion of the lunar satellite is as follows:

$$\left.\begin{aligned} \ddot{x} - 2n'\dot{y} - n'^2\left(x - \frac{M_1 c}{M + M_1}\right) &= \frac{\partial}{\partial x}(U + V), \\ \ddot{y} + 2n'\dot{x} - n'^2 y &= \frac{\partial}{\partial y}(U + V), \\ \ddot{z} &= \frac{\partial}{\partial z}(U + V), \end{aligned}\right\} \tag{5.2}$$

where V and U represent the terrestrial and the lunar gravitation potentials. These potentials can be expanded in series of Legendre polynomials (see § 4, Chapter 6 and § 3, Chapter 2):

$$V = \frac{fM_1}{c^2}x + \frac{fM_1}{2c^3}(2x^2 - y^2 - z^2) + \frac{fM_1}{c}\sum_{s=3}^{\infty}\left(\frac{r}{c}\right)^s P_s\left(\frac{x}{r}\right), \tag{5.3}$$

where r is the selenocentric distance of the satellite, $P_s(x/r)$ is a Legendre polynomial of the s-th order,

$$U = \frac{fM}{r} + f\sum_{s=2}^{\infty}\frac{Q_s(x, y, z)}{r^{2s+1}}, \tag{5.4}$$

[1]Assumption 4 is really quite close to reality, since the inclination of the earth's rotational axis to the plane of the lunar orbit amounts to 85°59'.

where $Q_s(x, y, z)$ are harmonic polynomials with respect to x, y, z of power s, the coefficients of which are linear combinations of the moments of inertia of the s-th order.

We now introduce the following designations:

$$U_1 = f \sum_{s=2}^{\infty} \frac{Q_s}{r^{2s+1}}, \tag{5.5}$$

$$V_1 = \frac{fM_1}{c} \sum_{s=3}^{\infty} \left(\frac{r}{c}\right)^s P_s\left(\frac{x}{r}\right). \tag{5.6}$$

We now introduce the new independent variable τ, which is related to t as follows: /263

$$\tau = (n - n')\, t, \tag{5.7}$$

in which n is the mean sidereal motion of the satellite. Introducing the designations

$$m = \frac{n'}{n - n'}, \quad \sigma = \frac{M_1}{M + M_1}, \quad k = \frac{fM}{(n - n')^2} \tag{5.8}$$

instead of the system (5.2), we obtain the following

$$\left.\begin{aligned} x'' - 2my' - m^2(1 + 2\sigma)x + \frac{kx}{r^3} &= \frac{1}{(n - n')^2}\frac{\partial}{\partial x}(U_1 + V_1), \\ y'' + 2mx' - m^2(1 - \sigma)y + \frac{ky}{r^3} &= \frac{1}{(n - n')^2}\frac{\partial}{\partial y}(U_1 + V_1), \\ z'' + m^2\sigma z + \frac{kz}{r^3} &= \frac{1}{(n - n')^2}\frac{\partial}{\partial z}(U_1 + V_1). \end{aligned}\right\} \tag{5.9}$$

Instead of the system (5.9) let us consider the following equations:

$$\left.\begin{aligned} x'' - 2my' - \frac{3}{2}m^2x + \frac{kx}{r^3} &= -\frac{1}{2}\mu\nu(1 - 4\sigma)x + \mu\frac{\partial}{\partial x}(\overline{U}_1 + \overline{V}_1), \\ y'' + 2mx' - \frac{3}{2}m^2y + \frac{ky}{r^3} &= -\frac{1}{2}\mu\nu(1 + 2\sigma)y + \mu\frac{\partial}{\partial y}(\overline{U}_1 + \overline{V}_1), \\ z'' + \sigma m^2 z + \frac{kz}{r^3} &= \mu\frac{\partial}{\partial z}(\overline{U}_1 + \overline{V}_1), \end{aligned}\right\} \tag{5.10}$$

in which we have made use of the following designations:

$$\left.\begin{aligned} \overline{U}_1 &= \frac{f}{(n-n')^2}\sum_{s=2}^{\infty}\frac{\overline{Q}_s}{r^{2s+1}}, \\ \overline{V}_1 &= \sigma r^2\sum_{s=1}^{\infty}\left(\frac{r}{c}\right)^s P_{s+2}\left(\frac{x}{r}\right), \end{aligned}\right\} \tag{5.11}$$

where $\overline{Q}_s$ represents the polynomials obtained from Q_s upon substitution of the following quantities in place of the moments of inertia J_s:

$$\overline{J}_s = J_s\mu^{-1}.$$

The initial system is obtained from (5.10) for $\mu = m^2/\nu$. The system of equations (5.10) for $\mu = 0$ has a partial solution: /264

$$x = a\cos\tau,\ y = a\sin\tau,\ z = 0, \tag{5.12}$$

where a is the root of the equation

$$ka^{-3} = 1 + 2m + \tfrac{3}{2}m^2 = l. \tag{5.13}$$

The circular motion described by (5.12) will not be Keplerian, but for small values of m it will deviate only very slightly from the latter.

Performing the transformation

$$\left.\begin{aligned} x &= a(1+\xi)\cos\tau - a\eta\sin\tau, \\ y &= a(1+\xi)\sin\tau + a\eta\cos\tau, \\ z &= a\zeta. \end{aligned}\right\} \tag{5.14}$$

we represent the system (5.10) in the following form:

$$\left.\begin{aligned} &\xi'' - 2(1+m)\eta' - l(1+\xi) + \frac{l(1+\xi)}{\rho^3} = \\ &\quad = -\frac{1}{2}\mu\nu[(1-\sigma)(1+\xi) - 3\sigma(1+\xi)\cos 2\tau + \\ &\qquad + 3\sigma\eta\sin 2\tau] + \mu\frac{\partial}{\partial\xi}(\overline{U}_1 + \overline{V}_1), \\ &\eta'' + 2(1+m)\xi' - l\eta + \frac{l\eta}{\rho^3} = -\frac{1}{2}\mu\nu[3\sigma(1+ \\ &\quad + \xi)\sin 2\tau + (1-\sigma)\eta + 3\sigma\eta\cos 2\tau] + \\ &\qquad + \mu\frac{\partial}{\partial\eta}(\overline{V}_1 + \overline{U}_1), \\ &\ddot{\zeta} + m^2\sigma\zeta + \frac{l\zeta}{\rho^3} = \mu\frac{\partial}{\partial\zeta}(\overline{U}_1 + \overline{V}_1), \end{aligned}\right\} \tag{5.15}$$

where

$$\rho^2 = (1 + \xi)^2 + \eta^2 + \zeta^2.$$

The solution of (5.12) for $\mu = 0$ corresponds to the trivial solution of the system (5.15). Assuming that this solution is generating, and using the Poincaré method, we will search for solutions of the system (5.15) which possess a period of 2π with respect to τ. With this purpose in mind, let us consider the equation of first approximation: /265

$$\left.\begin{aligned} \xi'' - 2(1+m)\eta' - 3l\xi &= -\tfrac{1}{2}\nu[(1-\sigma-3\sigma\cos 2\tau] + \\ &\quad + \frac{\partial}{\partial\xi}(\overline{U}_{11} + \overline{V}_{11}), \\ \eta'' + 2(1+m)\xi' &= -\tfrac{3}{2}\nu\sigma\sin 2\tau + \frac{\partial}{\partial\eta}(\overline{U}_{11} + \overline{V}_{11}), \\ \zeta'' + (l + m^2\sigma)\zeta &= \frac{\partial}{\partial\zeta}(\overline{U}_{11} + \overline{V}_{11}), \end{aligned}\right\} \qquad (5.16)$$

where the symbols $\overline{U}_{11}$ and $\overline{V}_{11}$ denote the sets of terms of the expansions of (5.11) which are linearly dependent upon ξ, η, ζ. The general solution of the system (5.16) is written as follows:

$$\left.\begin{aligned} \xi &= \beta_1\cos g\tau + \beta_2\sin g\tau + \frac{1}{g^2}[b + 2(1+m)\beta_5] + F_1(\tau), \\ \eta &= -\frac{2(1+m)\beta_1}{g}\sin g\tau + \frac{2(1+m)\beta_2}{g}\cos g\tau - \\ &\quad - [3l\beta_5 + 2(1+m)b]\frac{\tau}{g^2} + \beta_6 + F_2(\tau), \\ \zeta &= \beta_3\cos\omega\tau + \beta_4\sin\omega\tau + F_3(\tau), \end{aligned}\right\} \qquad (5.17)$$

where b is a quality which depends upon σ, ν, c and also upon the constants which characterize the shape of the moon; and the functions $F_i(\tau)$ are periodic with respect to τ, having a period of 2π. The quantities g and ω are defined as follows:

$$\omega^2 = l + m^2\sigma, \quad g^2 = 1 + 2m - \frac{m^2}{2}, \qquad (5.18)$$

while $\beta_1, \beta_2, \ldots, \beta_6$ are the constants of integration.

The conditions of periodicity (1.33) in our case as the following form:

$$\left.\begin{aligned}
\psi_1 &= \beta_1(\cos 2\pi g - 1) + \beta_2 \sin 2\pi g + \cdots = 0,\\
\psi_2 &= -\beta_1 g \sin 2\pi g + \beta_2 g(\cos 2\pi g - 1) + \cdots = 0,\\
\psi_3 &= \beta_3(\cos 2\pi\omega - 1) + \beta_4 \sin 2\pi\omega + \cdots = 0,\\
\psi_4 &= -\beta_3\omega \sin 2\pi\omega + \beta_4\omega(\cos 2\pi\omega - 1) + \cdots = 0,\\
\psi_5 &= -\frac{2(1+m)\beta_1}{g}\sin 2\pi g + \frac{2(1+m)\beta_2}{g}(\cos 2\pi g - 1) + \cdots = 0,\\
\psi_6 &= -2(1+m)\beta_1(\cos 2\pi g - 1) + 2(1+m)\beta_2 \sin 2\pi g + \cdots = 0.
\end{aligned}\right\} \qquad (5.19)$$

The Jacobian of the system (5.19) with respect to the quantities β_i, /266

$$\left[\frac{D(\psi_1, \psi_2, \ldots, \psi_6)}{D(\beta_1, \beta_2, \ldots, \beta_6)}\right]_0$$

is identically equal to zero; but since the system (5.2) admits of a Jacobi integral, then one of the equations of (5.19) is a corollary of the remaining ones. Therefore, according to (1.38) - (1.39), the question of the existence of periodic solutions is resolved by an examination of any five of the equations of (5.19). It is not difficult to show that the Jacobian of the first five equations will be as follows:

$$\left[\frac{D(\psi_1, \psi_2, \ldots, \psi_5)}{D(\beta_1, \beta_2, \ldots, \beta_5)}\right]_0 = \frac{96l\omega}{g}\sin^2\pi\omega \sin^2\pi g \qquad (5.20)$$

from which it is evident that the Jacobian becomes zero only upon the condition that at least one of the quantities g or ω is a whole number. This means that only for the denumerable set of solution belonging to the two-parameter family of generating periodic solutions, is the question of periodic solutions of system (5.2) unclear. For ∞^2 of generating periodic solutions, there exist a similar solution with the same period as that of the system (5.2).

§ 6. Periodic Solutions of the Limited Circular Three-Body Problem

Poincaré applied his method for the construction of periodic solutions to the limited circular three-body problem, using as generators the orbits of the two-body problem. In this way Poincaré and his successors were able to discover a number of classes of periodic orbits in this problem. At this point we shall consider one of these classes, for which the generating orbits are the orbits of the problem of two immobile centers.

It should be noted that the reasoning advanced in this section is justified not only for the circular three-body problem, but also for the problem of the motion of a satellite of a slowly rotating "triaxial" planet, as one can readily conclude by comparing the equations of motion of these two

problems (see formulas (1.24) - (1.26), Chapter 1).

In accordance with (1.24) - (1.26), the equations of motion of the plane circular three-body problem are written as follows: /267

$$\left.\begin{aligned} \ddot{x} &= 2n\dot{y} + U'_x, \\ \ddot{y} &= -2n\dot{x} + U'_y, \end{aligned}\right\} \tag{6.1}$$

where the forced function U is defined as follows:

$$U = \frac{n^2}{2}(x^2 + y^2) + \frac{fm_1}{r_1} + \frac{fm_2}{r_2}. \tag{6.2}$$

The system (6.1) admits of a first integral (Jacobi integral):

$$\dot{x}^2 + \dot{y}^2 = 2(U + h). \tag{6.3}$$

We now perform the regularizing transformation

$$\left.\begin{aligned} x &= c\left(\operatorname{ch} v \cos u - \frac{m_1 - m_2}{m_1 + m_2}\right), \\ y &= -c\operatorname{sh} v \sin u, \\ dt &= n'(\operatorname{ch}^2 v - \cos^2 u)\, d\tau, \end{aligned}\right\} \tag{6.4}$$

in which c denotes half the distance between the gravitating points, and n' denotes the mean motion of the passively gravitating point along an unperturbed (Keplerian) orbit. We shall consider here only those forms of motion for which

$$\mu = \frac{n}{n'}$$

is quite small.

As a result of the transformation (6.4) we obtain the following system of equations:

$$\left.\begin{aligned} u'' &= 2\mu(\operatorname{ch}^2 v - \cos^2 u)\, v' + W'_u, \\ v'' &= -2\mu(\operatorname{ch}^2 v - \cos^2 u)\, u' + W'_v, \end{aligned}\right\} \tag{6.5}$$

where the force function W has the following form:

$$W = \frac{f(m_1+m_2)}{n'^2c^3}\operatorname{ch} v + \frac{f(m_1-m_2)}{n'^2c^3}\cos u + \\ + \frac{h}{2n'^2c^3}(\operatorname{ch} 2v - \cos 2u) + \frac{\mu^2}{16c^2}(\operatorname{ch} 4v - \cos 4u) - \\ - \frac{\mu^2(m_1-m_2)}{4c^2(m_1+m_2)}(\cos u \operatorname{ch} 3v - \cos 3u \operatorname{ch} v). \tag{6.6}$$

The Jacobi integral in terms of the new variables is written as follows: /268

$$u'^2 + v'^2 - 2W = 0. \tag{6.7}$$

For $\mu = 0$, equations (6.5) represent the equations of motion of the problem of two immobile centers (see formulas (2.24) - (2.25) Chapter 3), and are integrated in quadratures. These solutions will be assumed to be generating.

Using the Poincaré method, we shall look for periodic solutions of the system (6.5) in the form of power series of the parameter μ:

$$u = \sum_{k=0}^{\infty} \mu^k u_k, \quad v = \sum_{k=0}^{\infty} \mu^k v_k. \tag{6.8}$$

For the sake of simplicity we shall assume that the moving point intersects the line of centers at the moment $\tau = 0$, at which moment the point is located at the pericenter or else at the apocenter of the generating orbit.

The system (6.5) is invariant with respect to the substitution

$$v \to v, \; u \to -u, \; \tau \to -\tau. \tag{6.9}$$

Therefore, by reason of the symmetry theorem, the conditions of periodicity of the solution with initial conditions

$$v(0) = v_0, \; v'(0) = 0, \; u(0) = 0, \; u'(0) = u_0' \tag{6.10}$$

can be transformed to the following form:

$$u(0) = 0, \quad u\left(\frac{T}{2}\right) = 0, \quad v'(0) = 0, \quad v'\left(\frac{T}{2}\right) = 0. \tag{6.11}$$

From (6.4), (6.9) and (6.10) it is evident that the conditions of periodicity (6.11) enable us to establish the existence of periodic orbits which are symmetrical with respect to the x-axis and which intersect that axis at a

right angle[1].

We shall use the symbols $u_0(\tau)$, $v_0(\tau)$ to denote the solution of the simplified system; and we shall write the system of equations of the first approximation as follows: /269

$$u_1'' + \left[\frac{f(m_1 - m_2)}{n'^2 c^3}\cos u_0 - \frac{h}{2n'^2 c^3}\cos 2u_0\right] u_1 = 2(\operatorname{ch}^2 v_0 - \cos^2 u_0)\, v_0', \tag{6.12}$$

$$v_1'' + \left[\frac{f(m_1 + m_2)}{n'^2 c^3}\operatorname{ch} v_0 + \frac{h}{2n'^2 c^3}\operatorname{ch} 2v_0\right] v_1 = -2(\operatorname{ch}^2 v_0 - \cos^2 u_0)\, u_0'. \tag{6.13}$$

Formulas (2.22), which determine the solution of the equations in variations for quasi-Liouville systems, afford a general solution for equations (6.12) and (6.13) in the following form:

$$u_1 = u_0'\left\{\beta_1\int\frac{d\tau}{u_0'^2} + \beta_2 + \int\left[\int(\operatorname{ch} 2v_0 - \cos 2u_0)\, u_0' v_0' d\tau\right]\frac{d\tau}{u_0'^2}\right\}, \tag{6.14}$$

$$v_1 = v_0'\left\{\beta_3\int\frac{d\tau}{v_0'^2} + \beta_4 + \int\left[\int(\operatorname{ch} 2v_0 - \cos 2u_0)\, u_0' v_0' d\tau\right]\frac{d\tau}{v_0'^2}\right\}. \tag{6.15}$$

Let us consider the simplified system of equations:

$$u_0'' = -\frac{f(m_1 - m_2)}{n'^2 c^3}\sin u_0 + \frac{h}{n'^2 c^3}\sin 2u_0, \tag{6.16}$$

$$v_0'' = \frac{f(m_1 + m_2)}{n'^2 c^3}\operatorname{sh} v_0 + \frac{h}{n'^2 c^3}\operatorname{sh} 2v_0 \tag{6.17}$$

and let us assume that the initial conditions correspond to the type of motion considered in § 2, Chapter 3. Then the generating solution is determined by formulas (2.47) and (2.56) of the section just referred to:

$$\cos u_0 = \frac{\operatorname{cn}^2[\sigma_1(\tau - \tau_1), k_1] - \beta^2}{\operatorname{cn}^2[\sigma_1(\tau - \tau_1), k_1] + \beta^2}, \tag{6.18}$$

$$\operatorname{ch} v_0 = \frac{1 + a^2\operatorname{cn}^2(\sigma\tau, k)}{1 - a^2\operatorname{cn}^2(\sigma\tau, k)}, \tag{6.19}$$

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[1] The equations of motion are also invariant with respect to the substitution $u \to u$, $v \to -v$, $\tau \to -\tau$. It follows from this that the conditions of periodicity may be written in the form $u'(0) = 0$, $u'(T/2) = 0$, $v(0) = 0$ $v(T/2) = 0$. With the help of these conditions of periodicity, it is possible to find periodic orbits both of direct and of indirect synodic motion.

where /270

$$\tau_1 = -K(k_1)/\sigma_1.$$

Since we are chosing a periodic generating solution,

$$\sigma_1 K(k) = s\sigma K(k_1), \tag{6.20}$$

where s is an arbitrary whole number, while K(k) and $K(k_1)$ are complete elliptic integrals of the first type. The period of the generating solution can be represented as follows:

$$T = \frac{2pK(k_1)}{\sigma} = \frac{2qK(k)}{\sigma_1}, \tag{6.21}$$

where p and q are mutually prime whole numbers.

From (6.18) and (6.19), we find that

$$u_0' = \frac{2\beta\sigma_1 \operatorname{sn}\sigma_1(\tau-\tau_1)\operatorname{dn}\sigma_1(\tau-\tau_1)}{\operatorname{cn}^2\sigma_1(\tau-\tau_1)+\beta^2}, \tag{6.22}$$

$$v_0' = -\frac{2a\sigma \operatorname{sn}\sigma\tau \operatorname{dn}\sigma\tau}{1-a^2\operatorname{cn}^2\sigma\tau}. \tag{6.23}$$

Let us calculate the integral

$$I = 4\beta^2\sigma_1^2 \int \frac{d\tau}{u_0'^2}. \tag{6.24}$$

From (6.22) and (6.24) we find that

$$I = \int \frac{\beta^2 + \operatorname{cn}^2\sigma_1(\tau-\tau_1)}{\operatorname{sn}^2\sigma_1(\tau-\tau_1)\operatorname{dn}^2\sigma_1(\tau-\tau_1)}\, d\tau. \tag{6.25}$$

With the help of the basic identities which associate the elliptic Jacobi functions, we transform (6.25) to the following form

$$I = -\frac{1}{k_1^2}\int d\tau + (1+\beta^2)^2 \int \frac{d\tau}{\operatorname{sn}^2\sigma_1(\tau-\tau_1)} + \frac{g_1}{k_1^2}\int \frac{d\tau}{\operatorname{dn}^2\sigma_1(\tau-\tau_1)}, \tag{6.26}$$

where $q_1 = (1+\beta^2)\,k_1^2 - 1$. Making use of the recurrent relationships /271

$$\int \operatorname{sn}^m z\, dz = \frac{1}{m+1}\left[\operatorname{sn}^{m+1} z \operatorname{cn} z \operatorname{dn} z + (m+2)(1+k^2)\int \operatorname{sn}^{m+2} z\, dz - (m+3)k^2 \int \operatorname{sn}^{m+4} z\, dz\right], \tag{6.27}$$

$$\int \mathrm{dn}^m z\,dz = \frac{1}{(m+1)k'^2}\Big[k^2\,\mathrm{dn}^{m+1}z\,\mathrm{sn}\,z\,\mathrm{cn}\,z + \\ + (m+2)(2-k^2)\int \mathrm{dn}^{m+2}z\,dz - (m+3)\int \mathrm{dn}^{m+4}z\,dz\Big], \tag{6.28}$$

instead of (6.26) we arrive at

$$I = -\frac{\tau}{k_1^2} - \frac{(1+\beta^2)\,\mathrm{cn}\,\sigma_1(\tau-\tau_1)\,\mathrm{dn}\,\sigma_1(\tau-\tau_1)}{\sigma_1\,\mathrm{sn}\,\sigma_1(\tau-\tau_1)} + \\ + \frac{g_1^2\,\mathrm{sn}\,\sigma_1(\tau-\tau_1)\,\mathrm{cn}\sigma_1(\tau-\tau_1)}{\mathrm{dn}\,\sigma_1(\tau-\tau_1)} + \frac{k_1^2(1+\beta^2)^2}{\sigma_1}\int \mathrm{sn}^2\sigma_1(\tau-\tau_1)d(\sigma_1\tau) + \\ + \frac{k_1^2}{\sigma_1 k_1^2 k_1'^2}\int \mathrm{dn}^2\,\sigma_1(\tau-\tau_1)\,d(\sigma_1 t). \tag{6.29}$$

But since

$$\int \mathrm{sn}^2 z\,dz = \frac{1}{k^2}[z - \mathrm{E}(\mathrm{am}\,z, k)], \tag{6.30}$$

$$\int \mathrm{dn}^2 z\,dz = \mathrm{E}(\mathrm{am}\,z, k), \tag{6.31}$$

where E is an elliptic integral of second type, then instead of (6.29) we shall finally arrive at

$$I = f_1\tau - \frac{(1+\beta^2)^2\,\mathrm{cn}\,\sigma_1(\tau-\tau_1)\,\mathrm{dn}\,\sigma_1(\tau-\tau_1)}{\sigma_1\,\mathrm{sn}\,\sigma_1(\tau-\tau_1)} - \\ - \frac{g_1^2\,\mathrm{sn}\,\sigma_1(\tau-\tau_1)\,\mathrm{cn}\,\sigma_1(\tau-\tau_1)}{\sigma_1\,\mathrm{dn}\,\sigma_1(\tau-\tau_1)} + f_2\mathrm{E}[\mathrm{am}\,\sigma_1(\tau-\tau_1), k_1], \tag{6.32}$$

in which

$$f_1 = \frac{g_1}{k_1^2}, \tag{6.33}$$

$$f_2 = \frac{1}{\sigma_1}\left[\frac{g_1^2}{k_1^2 k_1'^2} - (1+\beta^2)^2\right]. \tag{6.34}$$

By reasons of (6.32) the first term in (6.18) will be finite and continuous. /272
A similar investigation can be made for the second integral in (6.18).

And so,

$$u_1 = \frac{\beta_1}{2\beta\sigma_1}\left\{\left[f_1\tau - \frac{g_1^2\,\mathrm{sn}\,\sigma_1(\tau-\tau_1)\,\mathrm{cn}\,\sigma_1(\tau-\tau_1)}{\sigma_1 k_1'^2\,\mathrm{dn}\,\sigma_1(\tau-\tau_1)} + \right. \\ \left. + f_2\mathrm{E}(\mathrm{am}\,\sigma_1(\tau-\tau_1), k_1)\right]\frac{\mathrm{sn}\,\sigma_1(\tau-\tau_1)\,\mathrm{dn}\,\sigma_1(\tau-\tau_1)}{\mathrm{cn}^2\,\sigma_1(\tau-\tau_1)+\beta^2} -$$

$$- \frac{(1+\beta^2)^2 \operatorname{cn}\sigma_1(\tau-\tau_1)\operatorname{dn}\sigma_1(\tau-\tau_1)}{\operatorname{cn}^2\sigma_1(\tau-\tau_1)+\beta^2}\Big\} +$$
$$+ \frac{2\beta\beta_2\sigma_1 \operatorname{sn}\sigma_1(\tau-\tau_1)\operatorname{dn}\sigma_1(\tau-\tau_1)}{\operatorname{cn}^2\sigma_1(\tau-\tau_1)+\beta^2} +$$
$$+ u_0' \int \Big[\int (\operatorname{ch} 2v_0 - \cos 2u_0)\, u_0' v_0'\, d\tau\Big] \frac{d\tau}{u_0'^2}. \qquad (6.35)$$

Let us examine the first two conditions of periodicity (6.11). Let p be an odd number. Then these conditions may be written as follows:

$$\left.\begin{aligned} u_1\left(\frac{T}{2}\right) &= -\frac{1+\beta^2}{2\beta\sigma_1^2}\beta_1 + \ldots = 0, \\ u_1(0) &= \frac{k_1'\beta_1}{2\beta\sigma_1}\left[\frac{f_1}{\sigma_1}K(k_1) + f_2 E(k_1)\right] + \frac{2\sigma_1 k_1'}{\beta}\beta_2 + \ldots = 0. \end{aligned}\right\} \qquad (6.36)$$

From (6.36) it is evident that the Jacobian

$$\left[\frac{D\left[u_1(0),\ u_1\left(\frac{T}{2}\right)\right]}{D(\beta_1, \beta_2)}\right]_0 = \pm \frac{(1+\beta^2)k_1'}{\sigma_1\beta^2} \qquad (6.37)$$

is always different from zero. If p is an even number, then the first of the equations of (6.36) can be replaced with the following:

$$u_1\left(\frac{T}{2}\right) = \pm\frac{k_1'}{\sigma_1\beta^3}\left[\frac{q}{k_1^2}(p+1)K(k_1) + \sigma_1 f_2 q E(k_1)\right]\beta_1 \pm \frac{2\sigma_1 k_1'}{\beta}\beta_2 + \ldots = 0. \qquad (6.38)$$

From this we conclude that the corresponding Jacobian is not equal to zero. Therefore, with initial conditions which satisfy (6.20) periodic systems do exist. In the foregoing analysis no account was taken of the second of the integrals which appear in (6.12). It is easy to demonstrate that this integral will yield a secular term, which, upon the appropriate choice of β_1, will disappear, so that the solution will then be periodic for u. /273

Let us turn now to equation (6.13). By means of transformations analogous to the preceding, and with the help of recurrent formulas (6.27) and (6.28) we obtain the following expression for v_1:

$$v_1 = -\frac{\beta_3}{2a\sigma^2(1-a^2\operatorname{cn}\sigma\tau)}\Big\{\Big[(1-a^2)^2 - \frac{\sigma^4}{k^4}\Big]\sigma\tau \operatorname{sn}\sigma\tau \operatorname{dn}\sigma\tau -$$
$$-(1-a^2)^2 \operatorname{dn}^2\sigma\tau \operatorname{cn}\sigma\tau - \frac{(k^2+k'^2a^2)^2}{k'^2}\operatorname{sn}^2\sigma\tau \operatorname{cn}\sigma\tau +$$
$$+ \frac{(k^2+a^2k'^2)^2 - (1-a^2)k^2k'^2}{k^2k'^2} E(\operatorname{am}\sigma\tau, k)\cdot \operatorname{sn}^2\sigma\tau \operatorname{dn}\sigma\tau\Big\} - \beta_3 \frac{2a\sigma \operatorname{sn}\sigma\tau \operatorname{dn}\sigma\tau}{1-a^2\operatorname{cn}^2\sigma\tau} + \ldots \qquad (6.39)$$

Now, following differentiation with respect to τ (6.39), we can write the third and the fourth of the conditions of periodicity (6.11) in explicit form. The third condition is written as follows:

$$v_1'(0) = -\frac{2a\sigma^2}{1-a^2}\beta_4 + \ldots = 0. \tag{6.40}$$

The latter condition for an uneven value of q, will be

$$v_1'\left(\frac{T}{2}\right) = \pm\frac{\beta_3}{2a\sigma k'} + \ldots = 0. \tag{6.41}$$

From (6.40) and (6.41) we find that the Jacobian

$$\left[\frac{D\left(v_1'(0),\ v_1'\left(\frac{T}{2}\right)\right)}{D(\beta_3,\ \beta_4)}\right]_0 = \pm\frac{\sigma}{k'(1-a^2)} \tag{6.42}$$

is always different from zero. If q is an even number, then instead of (6.41) we must take the following expression:

$$v_1'\left(\frac{T}{2}\right) = \pm\frac{q\beta_3}{4a\sigma(1-a^2)}\left\{2\left[-\frac{a^4}{k^2} + (1-a^2)^2\right]K + \right.$$
$$\left. + \left[\frac{(k^2 + a^2k'^2)^2}{k^2k'^2} - (1-a^2)^2\,\Xi\right]\right\} \pm \frac{2a\sigma^2}{1-a^2}\beta_4 + \ldots = 0. \tag{6.43}$$

The coefficient for β_3 in (6.43) is different from zero, so that in the second case the corresponding Jacobian will not be equal to zero.

Thus, we have demonstrated the existence of periodic solutions of the initial equations which correspond to the generating solutions (6.18), (6.19). The solutions found will be periodic on the rotating plane, and they will close following several (many) revolutions. On an immobile plane these solutions will be near-periodic. /274

§ 7. Whittaker's Method. Near-Periodic Orbits of the Satellite of a Spheroidal Planet

Let us consider the motion of a satellite in the gravitational field of a spheroidal planet. We shall adopt a planetocentric coordinate system Oxyz, the basic plane of which coincides with the equatorial plane of the planet, while the z-axis coincides with the axis of rotation of the planet. The forced function of this problem will be formulated as follows:

$$U = \frac{fm}{r} + \mu R(\rho,\ z,\ \mu), \tag{7.1}$$

where

$$\rho^2 = x^2 + y^2,\ r^2 = \rho^2 + z^2.$$

In formula (7.1) we have made use of the following designations: m is the mass of the planet, μ is the small parameter on the order of the compression of the planet. As regards the function R, it is assumed that it is differentiable and that it possesses continuous partial derivatives of the first order. We shall also assume that the expansion of function R in Legendre polynomials begins with terms which are proportional to the cube of the inverse distance (see Chapter 2).

The differential equations of the problem are written as follows (see (3.13), Chapter 1):

$$\left.\begin{aligned} \ddot{\rho} - \rho\dot{\lambda}^2 &= \frac{\partial U}{\partial \rho}, \\ \frac{d}{dt}(\rho^2\dot{\lambda}) &= 0, \\ \ddot{z} &= \frac{\partial U}{\partial z}. \end{aligned}\right\} \tag{7.2}$$

In the system (7.2) the symbol λ denotes the planetocentric longitude of the satellite. With the help of the angular-momentum integral,

$$\rho^2\lambda = c, \tag{7.3}$$

in which c is an arbitrary constant, we can exclude from (7.2) the cyclic coordinate λ: /275

$$\ddot{\rho} = \frac{\partial W}{\partial \rho},\quad \ddot{z} = \frac{\partial W}{\partial z}, \tag{7.4}$$

where

$$W = -\frac{c^2}{2\rho^2} + U. \tag{7.5}$$

System (7.4) admits of an energy integral:

$$\dot{\rho}^2 + \dot{z}^2 = 2(W + h). \tag{7.6}$$

We shall demonstrate the existence of periodic solutions of system (7.4), by applying Whittaker's criterion[1] [160]. To do this it is necessary to

[1] See also Virchrw [161]. The suggested results were obtained by the author[162].

demonstrate that in the plane ρ, z there exists a ring domain, bounded by two closed curves, such that at all points on the inner boundary of which the expression

$$D = 2k(W + h) + \left(\frac{\partial W}{\partial \rho}\cos\gamma + \frac{\partial W}{\partial z}\sin\gamma\right) \tag{7.7}$$

is negative and at all points on the outer boundary, positive. In expression (7.7) k denotes the curvature of the corresponding boundary of the ring domain, while γ denotes the angle between the external normal to the boundary contour and the axis ρ.

Expression (7.7) has the following form:

$$D = 2kh - \frac{c^2}{\rho^2}\left(k - \frac{\cos\gamma}{\rho}\right) + \frac{fm}{r}\left(2k - \frac{\rho}{r^2}\cos\gamma - \frac{z}{r^2}\sin\gamma\right) + \\ + \mu\left(2kR + \frac{\partial R}{\partial \rho}\cos\gamma + \frac{\partial R}{\partial z}\sin\gamma\right). \tag{7.8}$$

Let us examine the ring domain which is bounded by the concentric circles

$$r = r_1, \quad r = r_2 \quad (r_1 < r_2). \tag{7.9}$$

Then the system of equations (7.4) admits of periodic solutions, provided it is possible to select r_1 and r_2 such that

$$D(r_1) < 0, \; D(r_2) > 0. \tag{7.10}$$

From (7.8) and (7.9) we find that /276

$$D(r_i) = \frac{2h}{r_i} + \frac{fm}{r_i^2} + \mu\left(\frac{2}{r_i}R + \frac{\rho}{r_i}\frac{\partial R}{\partial \rho} + \frac{z}{r_i}\frac{\partial R}{\partial z}\right) \quad (i = 1, 2). \tag{7.11}$$

In equations (7.11) ρ and z, respectively, satisfy the equations

$$\rho^2 + z^2 = r_i^2 \quad (i = 1, 2). \tag{7.12}$$

Studying the integral of (7.6) by the Hill method (see § 9, Chapter 3), it is possible to demonstrate that all the trajectories which are of practical interest will appear only when h < 0. It is precisely this case which we shall study here.

If μ = 0, then for any fixed value of h we can find values of r_1 and r_2 such that the conditions of (7.10) will be met. Then, by reason of a

continuity of the force function and its derivatives with respect to μ, for any values of h (given sufficiently small values of the parameter μ) we can also find values of r_1 and r_2 for which the inequalities (7.10) hold: in other words, for a spheroidal planet with sufficiently small compression, at least one periodic solution will be found among any particular isoenergetic family of solutions.

The existence of periodic solutions for all spheroidal bodies can be demonstrated if we take into account the characteristics of the gravitational potential exercised by the perturbing function R. In this case, however, it is sufficient to demonstrate the existence of periodic orbits well removed from the gravitating spheroidal body.

It is of interest that two periodic solutions of the system (7.4) there may correspond near-periodic motions defined by the system of equations (7.2). In this connection, let us assume that

$$\rho = \rho(t), \; z = z(t) \tag{7.13}$$

is a certain solution with period T.

From the integral (7.3) we have

$$\dot{\lambda} = \frac{c}{\rho^2(t)}, \tag{7.14}$$

whence

$$\lambda = \lambda_0 + c\int \frac{dt}{\rho^2(t)}. \tag{7.15}$$

The periodic function $1/\rho^2(t)$ can be represented in the form /277

$$\frac{1}{\rho^2} = a + \varphi(t), \tag{7.16}$$

where a is a certain constant, and ϕ is a certain periodic function satisfying the condition

$$\int_0^T \varphi(t)\,dt = 0. \tag{7.17}$$

From (7.15) and (7.16), we find that

$$\lambda = \lambda_0 + act + \Phi(t), \tag{7.18}$$

where $\Phi(t)$ is a function with the period T.

When the quantity t varies by an amount T, the coordinates ρ and z assume their initial values, while λ increases by an amount acT, which, generally speaking, will not be a multiple of 2π, since the period T and the constant a do not depend upon the area constant c. The motions will be periodic for the denumerable set of values of c, for which the quantity acT is a multiple of 2π.

For any given isoenergetic family of trajectories, there will exist ∞^3 near-periodic motions, of which ∞^2 will be periodic (here we take into account the arbitrary constant λ_0 which figures in equation (7.18).

Localization of near-periodic trajectories of the satellite can be accomplished with the method suggested by N. D. Moiseyev [163]; determining the period of the solution is possible with use of the results obtained by N. F. Reyn [164]. The iteration process can be carried out on electronic computers.

CHAPTER VII

THE STABILITY OF SATELLITE MOTIONS

§ 1. The Problem of Stability with Respect to a Certain Number of Variables. V. V. Rumyantsev's Theorem /278

In any expedient selection of orbits for artificial celestial bodies, we must consider not only the energy characteristics but also the "degree of sensitivity" of the trajectory, both to the initial deviations in generalized coordinates and velocities, and to small, constantly active perturbing forces. Just how substantial the latter factor may be can be judged from the effect which the moon's slight gravitational pull had on the American "Explorer-6" satellite: the satellite's life in this case was reduced from 20 years to two years.

Thus, in problems of celestial ballistics, the chosen solutions of the differential equations of motion of an artificial celestial body must, in one sense or another, possess the property of stability. In the majority of cases it is important to study the stability of solutions in the sense intended by A. M. Lyapunov; but in some cases trajectory stability in the sense of Lagrange [165] is all that need be considered. Also of interest in this connection is the stability of solutions as defined by V. I. Arnol'd [166-168].

We ascribe central significance to the study of stability in respect to a limited number of factors. This approach to the problem of stability for a long time escaped the attention of students of mechanics, and only comparatively recently was successfully developed by V. V. Rumyantsev, following which it found extensive application in a number of important problems of mechanics [169, 170, 171].

The property of stability of motion, at least in integrable problems of mechanics, is to one degree or another inherent in any solution. This is obvious from the following elementary considerations.

Let us take the variables "action-angle" as the canonical variables in an integrable problem of dynamics with n degrees of freedom (see § 10, Chapter 1). /279 In these variables the Hamiltonian system of differential equations of motion will have the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = 0 \quad (i = 1, 2, \ldots, n), \tag{1.1}$$

where the Hamiltonian H depends only upon the variables of action p_i. Then the general solution of the system (1.1) will be written as follows:

$$p_i = a_i, q_i = \omega_i t + b_i \ (i = 1, 2, \ldots, n), \tag{1.2}$$

where a_i and b_i are arbitrary constants, while

$$\omega_i = \frac{\partial H(a_1, a_2, \ldots, a_n)}{\partial a_i}.$$

From (1.2) it is evident that no matter how small the initial perturbations of the constants a_i may be, the canonical variables of "angle" type, following a certain finite interval of time, will differ from their unperturbed values by an indefinitely large amount. This means that with respect to variables of "angle" type the property of stability will be lacking. As regards canonical variables "action" type, the opposite is true -- they will differ from their unperturbed values by an indefinitely small amount, provided the initial perturbations are sufficiently small. This means that a solution of the system (1.1) is stable with respect to a limited number of factors -- more precisely, with respect to the quantities p_i. In the terms used in § 10, Chapter 1, we can say that for integrable Hamiltonian systems in a 2n-dimensional phase space q_i, p_i, perturbed motion will take place along the n-dimensional tori p_i = const, which, for any value of t, are fairly close to the unperturbed torus $p_i = a_i$ if the initial perturbations are fairly small. In the partial case, G. N. Duboshin [172] has arrived at a similar conclusion for the two-body problem.

It is necessary to remark, however, that in problems of dynamics we cannot usually expect to find stability with respect to all the variables; this is apparent from the example cited above, if we disregard equilibrium (in the broad sense) solutions. Therefore, in principle, the more correct formulation of the problem is that based on stability with respect to a limited number of variables. This formulation automatically resolves the problem (which Lyapunov leaves undecided) of whether it is always possible to construct differential equations of perturbed motion when the number of chosen quantities being studied in connection with stability is less than the number of degrees of freedom. /280

Formulation of the problem. Let us consider a system of differential equations in normal form,

$$\frac{dy_i}{dt} = Y_i(t, y_1, y_2, \ldots, y_n) \quad (i = 1, 2, \ldots, n) \tag{1.3}$$

and let us assume that any solution of this system can be extended indefinitely -- in other words, that it exists for any $t \geq t_0$, where t_0 is the initial moment. Let

$$y_i = f_i(t) \quad (i = 1, 2, \ldots, n) \tag{1.4}$$

be the partial solution of a system (1.3) which satisfies certain definite initial conditions:

$$t = t_0, \; y_i = y_i^0 \; (i = 1, 2, \ldots, n). \tag{1.5}$$

We shall refer to the solution (1.4) as "unperturbed", and any other solution as "perturbed". For a perturbed solution, instead of (1.5) the following initial conditions will be present:

$$t = t_0, \; y_i = y_i^{(0)} + \varepsilon_i \quad (i = 1, 2, \ldots n), \tag{1.6}$$

where ε_i represents certain real constants called initial perturbations.

For any $t \geq t_0$ the differences $y_i - f_i(t)$ will be referred to as "successive perturbations" or simply "perturbations". If all $\varepsilon_i = 0$, then all the differences $y_i - f_i(t) \equiv 0$.

Let $\Phi_i (i = 1, 2, \ldots, k \leq n)$ be assigned continuous functions $y_1, y_2, \ldots, \ldots, y_n$ of time t. The functions Φ_i, for any substitution of their values from (1.5) in place of y_i, become the familiar functions of time:

$$\Phi_i (t, \; f_1 (t), \ldots, f_n (t)) = \Phi_i^{(0)} (t) \quad (i = 1, \; 2, \ldots, k).$$

Let us consider the differences of the perturbed and the unperturbed values of the functions Φ_i: /281

$$x_i = \Phi_i (t, \; y_1, \ldots, y_n) - \Phi_i^{(0)} (t). \tag{1.7}$$

It is obvious that when every value of ε_i is zero, x_i will also be zero for any value of t. If the initial perturbations $x_i^{(0)} = x_i(t_0)$ are different from zero, and may assume any other sufficiently small absolute numerical value, then the perturbations of the functions Φ_i may always remain numerically fairly small, or else they may exceed any assigned small limits after a finite interval of time t. The solution of this problem depends both on the character of the right-hand members of system (1.3), and on the choice of the quantities Φ_i.

The solution of this problem is basic in the theory of stability.

Definition. An unperturbed solution is "stable" in Lyapunov's sense with respect to the quantities $\Phi_1, \Phi_2, \ldots, \Phi_k$ provided that for any number $\varepsilon > 0$

however small there will be found a number $\delta > 0$ such that for all real values $x_i^{(0)}$ which satisfy the condition

$$|x_i^{(0)}| \leqslant \delta \quad (i = 1, 2, \ldots, k), \tag{1.8}$$

the following inequality will be satisfied for any $t \geq t_0$:

$$|x_i(t)| < \varepsilon \quad (i = 1, 2, \ldots, k). \tag{1.9}$$

For all stable solutions, conditions (1.9) must be satisfied in the case of any initial perturbations consistent with (1.8). If there is found even one system of initial perturbations which, for any $t > t_0$, reduces to at least one of the inequalities of type

$$|x_i(t)| = \varepsilon$$

then the motion is "unstable".

Sometimes, of course, we are concerned not with the whole range of initial perturbations, but only with those which are subject to certain supplemental conditions of the type

$$f(x_1, x_2, \ldots, x_n) = 0 \tag{1.10}$$

or

$$f(x_1, x_2, \ldots, x_n) \geq 0. \tag{1.11}$$

Then, if the inequalities (1.9) are fulfilled for all initial perturbations satisfying conditions (1.8) and (1.10) (or (1.11)), the unperturbed solution in question is referred to as "conditional-stable". The stability of the solution, in the sense of the basic definition, we shall sometimes refer to as "unconditional". /282

If the perturbations of the quantities Φ_i, while satisfying the given definition, even for an indefinitely long period of time t, approach zero, then we say that the unperturbed motion is asymptotically stable with respect to the quantities $\Phi_1, \Phi_2, \ldots, \Phi_k$. In what follows we shall not formulate any criteria of asymptotic stability, since we are limiting ourselves to the study of the motion of a mechanical system in potential fields. As follows from Liouville's theorem [173], in this case the phase volume is incompressible, and, consequently, when definite limitations are imposed on the potential, asymptotic stability will not appear.

Now let us compile the differential equations of perturbed motion. We shall write (1.7) in the form

$$\psi_i = \Phi_i(t,\ y_1, \ldots, y_n) - \Phi_i^{(0)}(t) - x_i = 0 \qquad (1.12)$$

and assume that the functional determinant

$$\frac{D(\psi_1, \psi_2, \ldots, \psi_k)}{D(y_1, y_2, \ldots, y_k)}$$

is different from zero. We shall assume, also, that

$$x_i = y_i - f_i(t) \quad (i = k+1, \ldots, n) \qquad (1.13)$$

and we shall transform the system of equations of (1.3) to the variables x_1, $x_2, \ldots, x_n$, as defined by formulas (1.7) and (1.13). Differentiating these formulas with respect to time, we arrive at

$$\left.\begin{aligned} \frac{dx_i}{dt} &= \sum_{j=1}^{n} \frac{\partial \Phi_i}{\partial y_j} \dot{y}_j + \frac{\partial \Phi_i}{\partial t} - \frac{d\Phi_i^{(0)}}{dt} \quad (i = 1, 2, \ldots, k), \\ \frac{dx_i}{dt} &= \frac{dy_i}{dt} - \dot{f}_i(t) \quad (i = k+1, \ldots, n). \end{aligned}\right\} \qquad (1.14)$$

In place of $\dot{y}_i$ in the right-hand members of (1.14) we substitute expressions obtained from (1.3): /283

$$\left.\begin{aligned} \frac{dx_i}{dt} &= \sum_{j=1}^{n} \frac{\partial \Phi_i}{\partial y_j} Y_j + \frac{\partial \Phi_i}{\partial t} - \frac{d\Phi_i^{(0)}}{dt} \quad (i = 1, 2, \ldots, k), \\ \frac{dx_i}{dt} &= Y_i(t, y_1, \ldots, y_n) \quad (i = k+1, \ldots, n). \end{aligned}\right\} \qquad (1.15)$$

From formulas (1.7) and (1.13) we determine $y_1, y_2, \ldots, y_n$ (this is possible since the transformation Jacobian does not equal zero), and substitute the values found in the right-hand members of (1.15). Designating the result of the substitution by

$$X_i = X_i(t,\ x_1, \ldots, x_n) \quad (i = 1,\ 2, \ldots, n), \qquad (1.16)$$

in place of (1.15) we arrive at the following system of equations for perturbed motion:

$$\frac{dx_i}{dt} = X_i(t,\ x_1, \ldots, x_n) \quad (i = 1, 2, \ldots, n). \qquad (1.17)$$

It is clear from (1.16) that the functions $X_1, X_2, \ldots, X_n$ are identically equal to zero, provided all x_i are set equal to zero:

$$X_i(t, 0, 0, \ldots, 0) \equiv 0. \tag{1.18}$$

Consequently, the system (1.17) has a trivial solution,

$$x_1 = x_2 = \ldots = x_n = 0,$$

which corresponds to the solution (1.4) of the system (1.3). The problem of the stability of the solution of the system (1.3) with respect to the quantities Φ_i is equivalent to the problem of the stability of the trivial solution of the system (1.17) with respect to x_i ($i = 1, 2, \ldots, k$). When $k < n$, we study stability with respect to a portion of the variables. When $k = n$, the problem coincides with the Lyapunov formulation.

In studying the stability of the trivial solution of the system (1.17), we shall consistently assume that the right-hand members of (1.17) are holomorphic functions with respect to the variables $x_1, x_2, \ldots, x_n$ within a certain complex region of variation of the latter

$$|x_i| < A_i \quad (i = 1, 2, \ldots, n) \tag{1.19}$$

for all real values $t \geq t_0$. In addition, we shall assume that all X_i are continuous functions of time t when $t \geq t_0$.

Since basically we are going to make use of Lyapunov's second method, /284
we shall introduce still another concept of the Lyapunov function. Let us consider the function $V(t, x_1, x_2, \ldots, x_n)$, defined in the region of (1.19) for $t \geq t_0$. We shall assume that the function V is a differentiable function. Then its total derivative with respect to time is defined by the formula

$$\dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i + \frac{\partial V}{\partial t},$$

which, by reason of (1.17), can be written in the following form:

$$\dot{V} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} X_i + \frac{\partial V}{\partial t}. \tag{1.20}$$

The quantities stated in (1.20) is called "the derivative of the function V in virtue of the differential equations of perturbed motion".

Definition. If the function $V(t, x_1, x_2, \ldots, x_n)$ and its derivitive $\dot{V}$ in virtue of the differential equations are continuous and single-valued within the region of (1.19) for any $t \geq t_0$, and if, moreover, both V and $\dot{V}$ are identically equal to zero when $x_1 = x_2 = \ldots = x_n = 0$ for any $t \geq t_0$, then this particular function is called a Lyapunov function.

If a Lyapunov function is restricted to values which are either positive or negative (apart from zero), then it is said to be "constant-sine". If it is not so restricted, it is said to be "variable-sine".

Definition. If a constant-sine Lyapunov function which is independent of time $V(x_1, x_2, \ldots, x_n)$ in the region $|x_i| < A_i$, $i = 1, \ldots, k < n$, while x_{k+1}, $\ldots$, x_n are the derivatives, becomes zero only when the variables x_1, $x_2 \ldots$, x_k do so, then it is said to be "sine-limited with respect to the variables x_1, $x_2, \ldots, x_k$".

Definition. A Lyapunov function V, which is explicitly dependent upon time t, is said to be sine-limited with respect to the variables x_1, $x_2, \ldots, x_k$, if it is possible to find a positive function so defined which is not dependent upon time, $W(x_1, x_2, \ldots, x_k)$, such that the difference V - W, or -W - V is a positive function.

If, in the definitions quoted, k = n, then the function V will be sine-limited in the usual Lyapunov. /285

A. M. Lyapunov has supplied us with criteria of stability, asymptotic stability, and instability; these are contained in his basic theorems relating to the second method. A modification of these theorems has been given by V. V. Rumyantsev for the study of stability with respect to a limited number of variables. We shall prove the theorem relating to stability with respect to a limited number of variables simply by repeating the proof given by Rumyantsev [169, 170] without any significant changes.

Rumyantsev's theorem. If the differential equations of perturbed motion are such that we can find a function V which is sine-limited with respect to the variables x_1, $x_2, \ldots, x_k$, and whose derivative, in virtue of those equations is constant-sine with sine opposite to that of V or is identically equal to zero, then the unperturbed motion is constant with respect to the variables referred to.

Proof. Let $V(t, x_1, \ldots, x_n)$ be defined as positive with respect to x_1, $x_2, \ldots, x_k$, and let the time derivative (in virtue of the differential equations of motion (1.17)) of this function V be constant-sine-negative or else be identically equal to zero. Then, according to the definition of a time-dependent sine-limited function, it is possible to find a positive-defined

function $W(x_1, \ldots, x_k)$ such that $V - W \geq 0$.

Let us take an arbitrarily small number A, less than min $(A_1, A_2, \ldots, A_n)$. By $\mathcal{l} > 0$ we shall refer to the precise lowest limit of the function W on the sphere

$$\sum_{i=1}^{k} x_i^2 = A.$$

The function $V_0 = V(t_0, x_1, \ldots, x_n)$ is not explicitly dependent upon time, and therefore it admits of an infinitely small upper limit. (In Lyapunov's terminology, a bounded function $V(x_1, \ldots, x_n, t)$ admits of an infinitely small upper boundary when, for any arbitrarily small $\varepsilon > 0$, there can be found a number $h > 0$, such that, for all values of the variables satisfying the inequalities /286

$$t \geqslant t_0, \quad |x_i| \leqslant h,$$

the condition $|V| \leq \varepsilon$ will be fulfilled).

A value of $\mathcal{l}$ can be found, $\lambda > 0$, such that for all values of x_i subject to the condition $\sum_{i=1}^{n} x_i^2 \leq \lambda$, the following inequality will be fulfilled:

$$V_0 < \mathcal{l}. \tag{1.21}$$

If the initial perturbations $x_i^{(0)}$ of the quantities x_i satisfy the inequality

$$\sum_{i=1}^{n} x_i^{(0)2} \leqslant \lambda,$$

then from the equality

$$V - V_0 = \int_{t_0}^{t} \dot{V}\, dt$$

by virtue of (1.21) and the condition of the theorem $\dot{V} \leq 0$, we conclude that the values of the variables $x_1, \ldots, x_n$, as defined by the equations of perturbed motion (1.17), must always satisfy these conditions:

$$W \leq V \leq V_0 \leq \mathcal{l}.$$

However, since l is the precise lower limit of the function W on a sphere of radius A, the inequality

$$\sum_{i=1}^{k} x_i^2 < A,$$

will be fulfilled. The theorem is thereby proved.

Note. If the number of quantities x_i in connection with which we are studying stability is equal to n, then we obtain the basic theorem of Lyapunov regarding the stability of motion.

In a number of cases it is possible to establish not only the sufficient conditions for stability as given by the theorem which has been proved, but also the necessary conditions. For this purpose it is possible to make use of /287 certain theorems of Lyapunov regarding the equations of first approximation.

The holomorphic quality of the right-hand members of equations (1.17) makes it possible to represent the system (1.17) in the following form:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} p_{ij} x_j + X_i(x_1, \ldots, x_n, t), \tag{1.22}$$

where all p_{ij} are either real constants or known functions of time, while X_i are holomorphic functions of the quantities $x_1, \ldots, x_n$ whose expansions in powers of x_i begin with terms not lower than the second.

Let the equations (1.22) be such that all the coefficients p_{ij} are constant. In this case we speak of motion as being established in first approximation. However, if from equations (1.22) we throw out the functions of X_i, we thereby obtain a system of equations of first approximation which can be studied by a method familiar in the theory of differential equations [174]. The behavior of the solution of the equations of the first approximation depends upon the roots κ of the definitive (characteristic) equation,

$$D(\varkappa) = \det(\| p_{ij} \| - \varkappa E) = 0, \tag{1.23}$$

in which $||p_{ij}||$ is the matrix of coefficients, while E is the unit matrix.

Lyapunov demonstrated several theorems [174] with the help of which, in examining the equations of first approximation, it is possible to judge the stability of the trivial solution of system (1.22). We quote the theorems below without examining the proof.

Theorem 1. If the determining equation has roots only

with real negative terms, then the perturbed solution of system (1.22) is stable asymptotically, no matter how many functions X_i appear in the equations.

Theorem 2. If between the roots of the determining equation there are roots whose real terms are positive, then the unperturbed solution of system (1.22) is unstable, no matter how many functions X_i appear in the equations.

Theorem 3. If among the roots of the determining equation there are none with positive real terms, but there are some whose real terms are equal to zero, then the functions X_i can always be so chosen that the unperturbed solution of system (1.22) will be stable or unstable, as desired.

§ 2. The Method of N. G. Chetayev. Stability in the Presence of Constantly Acting Perturbations. /288

Routh [175] and later on Lyapunov [174] pointed out a method for the study of the stability of a solution when a sine-limited first integral exists for the equations of perturbed motion. If the sine-limited function is a first integral of the equations of perturbed motion, then the derivative of that function in virtue of the equations of motion is identically equal to zero; in this situation, according to the basic theorem of the preceding section, we know that the zero solution of equations (1.17) will be stable with respect to those variables for which the left-hand side of the first integral is sine-limited.

At the present time the method which N. G. Chetayev [176] proposed for constructing a Lyapunov function is being widely used. The essentials of the method are as follows. Let system (1.17) admit of s first integrals. None of which is a sine-limited function:

$$V_i(t, x_1, x_2, \ldots, x_n) = \text{const} \quad (i = 1, 2, \ldots, s). \tag{2.1}$$

We consider the function

$$V = f(V_1, V_2, \ldots, V_s, \lambda_1, \lambda_2, \ldots, \lambda_m), \tag{2.2}$$

which depends upon several indeterminant parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ so chosen that the function V will be sine-limited. This will occur whenever the expansion of the function V in series of increasing powers of x_i is begun with a homogeneous sine-limited form of even order with respect to the variables which are of interest. Usually the function (2.1) is a linear connective of the first integrals. If the constructed function is sine-limited, then the solution under consideration will be stable, by virtue of the basic theorem of the preceding section, since $\dot{V} \equiv 0$.

Effective use of first integrals in a different manner has been pointed out by Routh [175]. Let us consider a holonomic mechanical system with n degrees of freedom, the system being defined by the Lagrangian function $L(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$. Let the generalized coordinates $q_{k+1}, \ldots, q_n$ be cyclic, i.e., /289

$$\frac{\partial L}{\partial \dot{q}_i} = 0 \quad (i = k+1, \ldots, n). \tag{2.3}$$

The Lagrangians of the equation of motion of this mechanical system,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad (i = 1, 2, \ldots, n) \tag{2.4}$$

admit of n -k first integrals corresponding to cyclic coordinates:

$$\frac{\partial L}{\partial \dot{q}_i} = \beta_i \quad (i = k+1, \ldots, n). \tag{2.5}$$

For simplicity, we shall assume that perturbations have been taken as generalized coordinates (this does not limit the generality of the investigation). In other words, system (2.4) has a zero solution. Ignoring the cyclic coordinates and introducing the Routh function R (see § 2, Chapter 1), we have

$$R = L - \sum_{i=k+1}^{n} \beta_i \dot{q}_i, \tag{2.6}$$

in which the cyclic functions should be replaced with their values from the first integrals of (2.5) this gives us the equations of motion in Routhian form:

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0. \tag{2.7}$$

If the Routhian function is not explicitly dependent upon time, then system (2.7) admits of a first integral like the kinetic energy integral:

$$V = \sum_{i=1}^{k} \frac{\partial R}{\partial \dot{q}_i} \dot{q}_i - R = h. \tag{2.8}$$

According to Routh's theorem, if the function V is sine-limited with respect to the variables $q_1, q_2, \ldots, q_k$, the zero solution is stable on the assumption that the constants β_i of the cyclic integrals are not perturbing.

This is the way in which N. G. Chetayev [177] formulates the Routh theorem. /290
The correctness of the theorem of obvious: since $\dot{V} \equiv 0$, the stability follows directly from the basic theorem of § 1.

It is not difficult to demonstrate that the Routh theorem can be formulated in more general form. First of all, in the capacity of the Lyapunov function, we may select the connective of the first integrals, although not the generalized energy integral; secondly, it is not necessary to require sine-limitation of this connective with respect to all of the variables. Finally, there is a simple means which enables us to remove the condition of invariability of the quantities β_i, and to demonstrate unconditional stability with respect to a limited number of variables. The proof of this is based on the method employed by the present author in various articles [178, 179] (see also [17] and [180]).

Theorem. If the Lagrangians of equation (2.4) possess n - k cyclic integrals (2.5), and for certain values of the constants $\beta_i = \beta_i^{(0)}$ admit of a zero solution $q_1 = q_2 = \ldots = q_m = 0$, and if, moreover, for fixed values of β_i it is possible to construct a sine-limited (with respect to the variables $q_1, q_2, \ldots, q_m$ $(m \leq k)$ Lyapunov function $V(q_1, \ldots, q_m, \beta_{k+1}, \ldots, \beta_n)$, whose derivative in virtue of the differential equations of perturbed motion is identically equal to zero, then the zero solution is stable with respect to the quantities $q_1, \ldots, q_m, \beta_{k+1}, \ldots, \beta_n$ and is thereby unconditional.

Proof. Let us assume that with values $\beta_i = \beta_i^{(0)}$ of the arbitrary constants of the cyclic integrals (2.5), the system of equations (2.4) admits of a trivial solution

$$q_1 = q_2 = \ldots = q_m = 0,$$

and let us assume, moreover, that there exists a Lyapunov function

$$V = V(q_1, \ldots, q_m, \beta_{k+1}, \ldots, \beta_n), \tag{2.9}$$

which is sine-limited with respect to $q_1, \ldots, q_m$ for $\beta_i = \beta_i^{(0)}$, whose derivative satisfies the conditions of the theorem (with values $\beta_i \neq \beta_i^{(0)}$, the function V cannot, in general, be sine-limited in the indicated sense).

We stipulate that

$$q_i = x_i \quad (i = 1, 2, \ldots, k),$$
$$\beta_i - \beta_i^{(0)} = x_i \quad (i = k+1, \ldots, n).$$

Then, according to (2.6) and (2.7) the perturbed motion can be described by the following system of equations: /291

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{x}_i}\right) - \frac{\partial R}{\partial x_i} = 0 \quad (i = 1, 2, \ldots, k), \\ \frac{dx_i}{dt} = 0 \quad (i = k+1, \ldots, n). \end{aligned}\right\} \tag{2.10}$$

Expanding the function (2.9) in a power series of the perturbations $x_i = \beta_i - \beta_i^{(0)}$, we then have

$$V = V(x_1, \ldots, x_k, \beta_{k+1}^{(0)}, \ldots, \beta_n^{(0)}) + \sum_{i=k+1}^{n} \left(\frac{\partial V}{\partial \beta_i}\right)_{\beta_i = \beta_i^{(0)}} \cdot x_i + \cdots \tag{2.11}$$

By the condition of the function, (2.11) is sine-limited with respect to x_1, $x_2, \ldots, x_m$ when $\beta_i = \beta_i^{(0)}$.

Rejecting the assumption that the initial perturbations are subject to the conditions

$$\beta_i = \beta_i^{(0)},$$

we consider the system of equations (2.10). We then construct a new Lyapunov function $\bar{V}$ in the following form:

$$\bar{V} = V(x_1, \ldots, x_k, \beta_{k+1}^{(0)}, \ldots, \beta_n^{(0)}) + \sum_{i,\, j=k+1}^{n} A_{ij} x_i x_j, \tag{2.12}$$

where A_{ij} are constant quantities which should be chosen in such a way that the function $\bar{V}$ will be sine-limited not only with respect to $x_1, x_2, \ldots, x_m$, but also with respect to $x_{k+1}, \ldots, x_n$.

In order that the function (2.12) should be sine-limited, it is sufficient that the quadratic form $\Sigma A_{ij} x_i x_j$ should be sine-limited and positive with respect to $x_{k+1}, \ldots, x_n$. This can be easily achieved by the proper choice of the constants A_{ij}. Thus, with the appropriate choice of A_{ij}, the function (2.12) is positive-determined with respect to $x_1, x_2, \ldots, x_m, x_{k+1}, \ldots, x_n$. It is obvious that for a derivative of the function $\bar{V}$ in virtue of the differential equations of perturbed motion, we will have the following: /292

$$\dot{\overline{V}} = \dot{V} \equiv 0.$$

In accordance with the theorem of V. V. Rumyantsev, we conclude that the zero solution under consideration is stable with respect to the quantities x_1, x_2,... ..., x_m and x_{k+1},..., x_n for any values of the initial perturbations.

Let us proceed not to a consideration of stability in the presence of constantly acting perturbations. In constructing the differential equations of motion of the object under study, we shall begin in every instance on the basis of a certain simplified scheme, which does not take small forces into consideration. It is implicitly assumed that these disregarded forces, by reason of their smallness, do not exert any perceptable effect. In certain cases, of course, we know the structure of these small perturbing forces, and in certain others we do not. This means that the presence of Lyapunov stability in the solution of the system of differential equations under study does not, strictly speaking, justify the conclusion that the corresponding real motion will be stable: the theory of differential equations offers many examples in which small perturbing terms in the right-hand members of differential equations result in qualitative changes in the properties of the integral curves.

We must face the actual problem of investigating the stability of solutions of differential equations of motion in the presence of constantly active perturbing forces. Most frequently, in formulating this problem, only a single assumption is made regarding the character of the perturbing forces -- namely, that such forces do not, in absolute magnitude, exceed certain preassigned limits. The stability criteria which have been assigned for such a formulation impose rigid requirements on the properties of the simplified system of differential equations -- namely the system which does not contain constantly active perturbations. It appears [181] that the necessary condition for stability in the presence of constantly active perturbing forces is merely that the zero solution of the simplified system of equations should be uniformly asymptotically stable. But this requirement in the case of mechanical systems which are moving within potential force fields is never actually fulfilled, by reason of the existence of an integral invariant. For this reason the stability criterion referred to is frequently quite useless /293
in problems of dynamics.

However, in many situations it is possible to alter the formulation of the problem of stability in the presence of constantly acting perturbing forces, by imposing supplemental limitations upon those forces. An example of such an altered formulation may be found in the work of G. N. Duboshin [182], where the perturbing forces are assumed either to be constant, or to be linear functions of perturbations. But still another formulation of the problem is possible: we shall assume that we know the functional structure of the perturbing forces, although we do not have sufficiently precise knowledge of the numerical values of the parameters which enter into the analytical expressions of the forces. Such a situation occurs in actual practice in the study of the motion of a satellite within a central gravitational field, when it is known that the gravitational potential is in the form of a series in

spherical functions, but when the coefficients of the series have either been determined with a great degree of error, or else have not been determined at all but only estimated within the nearest order.

It is possible to impose upon the perturbing forces a less stringent limitation, requiring only that the differential equations of motion, written with allowance for the perturbing forces, shall retain a definite number of first integrals (either in unaltered or in slightly altered form). In this connection the method which N. G. Chetayev proposes for the construction of the Lyapunov function has often proved to be effective (see [178, 179]).

Let us construct the system of differential equations:

$$\frac{dx_i}{dt} = X_i(t, x_1, \ldots, x_n, \mu) \quad (i = 1, 2, \ldots, n), \tag{2.13}$$

which possesses s first integrals:

$$F_i(t, x_1, x_2, \ldots, x_n, \mu) = c_i \quad (i = 1, 2, \ldots, s). \tag{2.14}$$

We shall assume that the functions X_i and F_i are holomorphic with respect to $x_1, x_2, \ldots, x_n, \mu$ within a certain vicinity of the point $x_1 = x_2 = \ldots = x_n = \mu = 0$ for all values $t \geq t_0$. For convenience, without any loss in generality, we will assume that $\mu \geq 0$.

In addition, let the expansions in powers of the parameter μ of the right-hand members of equations (2.13) /294

$$X_i = \sum_{j=0}^{\infty} X_{ij}(t, x_1, x_2, \ldots, x_n)\,\mu^j \tag{2.15}$$

be such that

$$X_{i0}(t, 0, \ldots, 0) \equiv 0. \tag{2.16}$$

As regards the functions X_{ij} when $j \neq 0$, it should be remarked that in the general case they do not possess this property.

For $\mu = 0$, we obtain from (2.15) the following simplified system of differential equations:

$$\frac{dx_i}{dt} = X_{i0}. \tag{2.17}$$

System (2.15), by reason of the property of (2.16), has a zero solution:

$$x_1 = x_2 = \ldots = x_n = 0. \tag{2.18}$$

We shall accept (2.18) as the unperturbed solution, and shall investigate its stability on the assumption that both initial perturbations and constantly active perturbations are present. The latter, of course, are determined by the differences

$$X_i - X_{i0} = \sum_{j=1}^{\infty} \mu^j X_{ij}.$$

These difference may be arbitrary functions which satisfy a single requirement, namely the condition of the existence of s first integrals (2.14).

We shall assume also that the solution (2.18) is stable with respect to a certain number of the variables $x_1, x_2, \ldots, x_k$ ($k \leq n$), the property of stability being observed with the help of the connective of the first integrals (in the general case this is nonlinear):

$$V = V(F_{10}, F_{20}, \ldots, F_{s0}). \tag{2.19}$$

Here V is a sine-limited function with respect to the variables $x_1, \dot{x}_2, \ldots, x_k$, while $F_{i0} = F_i(t, x_1, \ldots, x_n, 0)$.

Let us examine the following system of differential equations of motion: /295

$$\left.\begin{aligned} \frac{dx_i}{dt} &= X_i(t, x_1, \ldots, x_n, x_{n+1}) \quad (i = 1, 2, \ldots, n), \\ \frac{dx_{n+1}}{dt} &= 0 \end{aligned}\right\} \tag{2.20}$$

This system admits of a zero solution which corresponds to the zero solution of system (2.17). We shall investigate the stability of this solution with respect to the quantities $x_1, \ldots, x_k, x_{n+1}$. If this solution turns out to be stable, we can on that basis reach the conclusion of the stability of the zero solution of the simplified system (2.17) in the presence of constantly active perturbations of the form of (2.18)(i.e., stability in the presence of perturbing forces of definite structure).

In order to study the stability in this case we shall construct a Lyapunov function in the following form:

$$\overline{V} = V(F_1, F_2, \ldots, F_s) + \gamma x_{n+1}^2, \tag{2.21}$$

where γ is a constant quantity subject to choice. Expanding $\overline{V}$ in power series of the small parameter μ, or, what amounts to the same thing, the quantity x_{n+1}, we arrive at

$$\overline{V} = V(F_{10}, \ldots, F_{s0}) + [\gamma + (V'_\mu)_0]x_{n+1}^2 + \ldots \tag{2.22}$$

If we also assume that the derivative $(V'_\mu)_0$ is bounded, then it is always possible to select the quantity γ in such a way that the signs of the first two terms will coincide. Then, given sufficiently small absolute values of the quantities $x_1, \ldots, x_k, x_{n+1}$, the sign of the function $\overline{V}$ will coincide with the sine of the first two components, and, therefore, the function $\overline{V}$ will be sine-limited with respect to the quantities $x_1, \ldots, x_k, x_{n+1}$[1].

Since, in addition, the derivative of $\overline{V}$ in virtue of equations (2.20) is identically equal to zero, it follows that the solutions (2.18) are stable with respect to a certain number of variables $x_1, \ldots, x_k, x_{n+1}$, in the presence of constantly active perturbations $X_i - X_{i0}$.

§ 3. The Stability of Circular Satellite Orbits

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Stability criteria for circular orbits in astronomical problems involved in the study of the motion of a material point within axisymmetrical gravitational fields have been given by G. N. Duboshin [183], S. Chandrasekhr [184], and N. G. Chetayev [177]. These criteria, however, have been established only for perturbances subject either to the condition of invariability of total mechanical energy, or to the condition of invariability of areal velocity. This leads only to the conditional stability of circular orbits. In [185] are stated the necessary and sufficient conditions for the stability of circular orbits of a point moving within an axisymmetrical potential field of forces. In the present section we shall arrive at a conclusion regarding the stability of circular orbits in the general case, following the procedure used in § 2 preceding.

The application of these results to problems of celestial ballistics was described in several works by the present author [89] and was subsequently developed by V. G. Degtyarev [186] and A. L. Kunitsyn [18].

Let us examine the motion of a material point in an axisymmetrical field of force. We shall choose our system of coordinates in such a way that the z-axis coincides with the axis of symmetry of the field, while the plane of the unperturbed orbit of the point serves as the basic coordinate plane. We

[1] Here and in what follows we shall omit elementary arguments dealing with the proof of the sine-limitation of the Lyapunov function, which is explicitly dependent upon time.

shall assume that the force function of the problem in cylindrical coordinates has the form $U = U(\rho\ z)$ and is a holomorphic function of ρ and z within a certain region of their variation, in which case $U(\rho, z) \equiv 0$ by reason of the choice of the basic coordinate plane. In addition, we shall assume that the condition

$$\left[\frac{\partial}{\partial\rho}\left(\rho^3\frac{\partial U}{\partial\rho}\right)\right]_0 \neq 0 \tag{3.1}$$

is satisfied at least for the circular orbits under consideration (here and subsequently the subscript "0" will signify that we have taken the value of the function at the point $\rho = \rho_0$, $z = 0$).

The differential equations of motion will be written as follows:

$$\ddot{\rho} = \frac{4\sigma^2}{\rho^3} + \frac{\partial U}{\partial\rho}, \quad \dot{\sigma} = 0, \quad \ddot{z} = \frac{\partial U}{\partial z}, \tag{3.2}$$

where σ is the areal velocity.

The system of equations (3.2) admits of a partial solution /297

$$\rho = \rho_0, \ \dot{\rho} = 0, \ \sigma = \sigma_0, \ z = 0, \ \dot{z} = 0, \tag{3.3}$$

to which corresponds a certain circular motion. The solution (3.3) exists, provided the force function of the problem satisfies the following inequality:

$$\left(\frac{\partial U}{\partial\rho}\right)_0 < 0. \tag{3.4}$$

The value of the areal velocity must satisfy the following equation:

$$\frac{4\sigma_0^2}{\rho_0^3} + \left(\frac{\partial U}{\partial\rho}\right)_0 = 0. \tag{3.5}$$

Let us consider the stability of the partial solution (3.3). Introducing the usual designations for perturbations,

$$x_1 = \rho - \rho_0, \ x_2 = \dot{\rho}, \ x_3 = \sigma - \sigma_0, \ x_4 = z, \ x_5 = \dot{z},$$

we write the differential equations of perturbed motion as follows:

$$\left.\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= 4(\sigma_0 + x_3)^2(\rho_0 + x_1)^{-3} + \frac{\partial U}{\partial x_1},\\
\dot{x}_3 &= 0,\\
\dot{x}_4 &= x_5,\\
\dot{x}_5 &= \frac{\partial U}{\partial x_4}.
\end{aligned}\right\} \qquad (3.6)$$

The system (3.6) possesses two integrals:

$$v_1 = x_3 = c', \qquad (3.7)$$

$$V_2 = \frac{1}{2}[x_2^2 + 4(\sigma_0 + x_3)^2(\rho_0 + x_1)^{-2} + x_5^2] - U(\rho_0 + x_1, x_4) = h. \qquad (3.8)$$

In order to obtain the necessary stability conditions of solution (3.3) with respect to the quantities ρ, $\dot{\rho}$, z, $\dot{z}$, we examine the equations of first approximation: /298

$$\left.\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= \left(U''_{\rho\rho} + \frac{3}{\rho}U'_\rho\right)_0 x_1 + \frac{4}{\rho_0^2}\sqrt{(-\rho U'_\rho)_0}\,x_3 + (U''_{\rho z})_0 x_4,\\
\dot{x}_3 &= 0,\\
\dot{x}_4 &= x_5,\\
\dot{x}_5 &= (U''_{\rho z})_0 x_1 + (U''_{zz})_0 x_4.
\end{aligned}\right\} \qquad (3.9)$$

The characteristic equation of system (3.9) has this form:

$$\lambda^4 - \left(U''_{\rho\rho} + \frac{3}{\rho}U'_\rho + U''_{zz}\right)_0 \lambda^2 + \left(U''_{\rho\rho} + \frac{3}{\rho}U'_\rho - U''^2_{\rho z}\right)_0 = 0. \qquad (3.10)$$

From the Lyapunov theorem given in § 1 it follows that a necessary stability condition is the absence, among the roots of the characteristic equation, of any roots which have positive real terms. In our case this condition will be met, provided the following inequality is satisfied:

$$\left.\begin{aligned}
\left(U''_{zz} + U''_{\rho\rho} + \frac{3}{\rho}U'_\rho\right)_0 &< 0,\\
\left[U''_{zz}\left(U''_{\rho\rho} + \frac{3}{\rho}U'_\rho\right) - U''^2_{\rho z}\right]_0 &> 0,
\end{aligned}\right\} \qquad (3.11)$$

or else one of the two following conditions, equivalent to it:

$$\left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)_0<0,\ \left[U''_{zz}\left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)-U''^{2}_{\rho z}\right]_0>0. \tag{3.12}$$

$$(U''_{zz})_0<0,\ \left[U''_{zz}\left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)-U''^{2}_{\rho z}\right]_0>0. \tag{3.13}$$

In the case of motion of the point within a force field with possesses symmetry with respect to the basic coordinate plane, the necessary stability conditions are particularly simple. Here, $(U''_{\rho z})_0 = 0$, so that in place of (3.11) we obtain the following necessary stability conditions:

$$\left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)_0<0,\ (U''_{zz})_0<0. \tag{3.14}$$

We should note that when the necessary stability conditions are satisfied, /299 the unperturbed motion under consideration will be stable with respect to ρ, $\dot{\rho}$, σ, z, $\dot{z}$ in first approximation.

Now let us proceed to a derivation of the sufficient stability conditions. Making use of N. G. Chetayev's method (see § 2), we select the connective of integrals (3.7) and (3.8) as the Lyapunov function:

$$V = V_2 + \lambda_1 V_1 + \lambda_2 V_1^2, \tag{3.15}$$

where λ_1 and λ_2 are arbitrary numbers which must be chosen in such a way that the function (3.15) will be sine-limited. The derivative of function V in virtue of the equations of perturbed motion (3.6) will be identically equal to zero. Therefore, with sine-limitation of function V, the solution (3.3) will be stable.

We now expand V in Taylor's series in the vicinity of the point $x_1 = x_2 = \ldots = x_5 = 0$:

$$\begin{aligned} V = &-\frac{1}{2}\left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)_0 x_1^2+\frac{1}{2}x_2^2+\left(\frac{2}{\rho_0^2}+\lambda_2\right)x_3^2+\\ &+\frac{1}{2}x_5^2-\frac{1}{2}(U''_{zz})_0 x_4^2-\frac{4}{\rho_0^2}\sqrt{(-\rho U'_{\rho})_0}\,x_1x_3-\\ &-(U''_{zz})_0 x_1x_4+\cdots \end{aligned} \tag{3.16}$$

First of all, selecting $\lambda_1 = -\frac{2}{\rho_0}\sqrt{(-\rho U')_0}$, we are able to eliminate the linear term from the expansion of (3.15). The function will be sine-limited, if the quadratic form which appears in the expansion (3.16) is positive definite. This happens provided the determinant

$$\begin{vmatrix} -\frac{1}{2}\left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)_0, & 0, & -\frac{2}{\rho_0^2}\sqrt{(-\rho U'_{\rho})_0}, & \frac{1}{2}(U''_{\rho z})_0, & 0 \\ 0, & \frac{1}{2}, & 0, & 0, & 0 \\ -\frac{2}{\rho_0^2}\sqrt{(-\rho U'_{\rho})_0}, & 0, & \lambda_2+\frac{2}{\rho_0^2}, & 0, & 0 \\ \frac{1}{2}(U''_{\rho z})_0, & 0, & 0, & -\frac{1}{2}(U''_{zz})_0, & 0 \\ 0, & 0, & 0, & 0, & \frac{1}{2} \end{vmatrix}$$

and provided that its principal diagonal minors are positive: i.e., the following inequalities are fulfilled: /300

$$\left.\begin{aligned} \left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)_0 &< 0, \\ -\frac{1}{4}\left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)_0\left(\lambda_2+\frac{2}{\rho_0^2}\right)+\frac{2}{\rho_0^3}(U'_{\rho})_0 &> 0, \\ \frac{1}{4}\left(\lambda_2+\frac{2}{\rho_0^2}\right)\left[U''_{zz}\left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)-U'^2_{\rho z}\right]_0- & \\ -\frac{2}{\rho_0^3}(U''_{zz}U'_{\rho})_0 &> 0. \end{aligned}\right\} \qquad (3.17)$$

It is not difficult to verify that if conditions (3.12) are met, and if there is an appropriate choice of λ_2, the inequalities (3.17) also hold. Consequently, the motion under consideration will be stable, while conditions (3.12) will be sufficient.

The stability conditions (3.12) are even more simple when the motion of the point in question takes place within a gravitational field. Using Laplace's (or Poisson's) equation, written in cylindrical coordinates (see § 1, Chapter 2),

$$\frac{\partial^2 U}{\partial z^2}+\frac{\partial^2 U}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial U}{\partial \rho}=0$$

(here it is assumed that the gravitational field possesses axial symmetry), one can express the derivative U''_{zz} in terms of derivatives with respect to ρ:

$$\frac{\partial^2 U}{\partial z^2}=-\frac{\partial^2 U}{\partial \rho^2}-\frac{1}{\rho}\frac{\partial U}{\partial \rho}. \qquad (3.18)$$

For the sake of simplicity we limit ourselves to the case of a gravitational field which is symmetrical with respect to the basic plane. We then

transform the stability conditions (3.14) to the following form:

$$\left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)_0<0,\ \left(U''_{\rho\rho}+\frac{3}{\rho}U'_{\rho}\right)_0>\left(\frac{2}{\rho}U'_{\rho}\right)_0, \tag{3.19}$$

or, following simple transformations, we have the following:

$$\left(2\rho^2\frac{\partial U}{\partial\rho}\right)_0<\left[\frac{\partial}{\partial\rho}(\rho^3U'_{\rho})\right]_0<0. \tag{3.20}$$

Making the additional assumption that the gravitational field is spherically symmetrical, we then obtain the following /301

$$(U''_{zz})_0=\left(\frac{1}{\rho}U'_r\right)_0<0,$$

where $r^2 = \rho^2 + z^2$. These inequalities are fulfilled whenever condition (3.4) holds: in other words, we arrive at the result that all existing circular orbits are stable.

Now let us apply the stability criteria which we have obtained to the problem of the motion of an artificial earth satellite within a normal gravitational field whose potential is defined by the following formula (see § 6, Chapter 3):

$$U=\frac{fM}{2}\left[\frac{1}{\sqrt{\rho^2+(z-ci)^2}}+\frac{1}{\sqrt{\rho^2+(z+ci)^2}}\right]. \tag{3.21}$$

Since the force function (3.21) is even with respect to z, circular motion of artificial earth satellites may take place within the equatorial plane z = 0. This requires that condition (3.4) be met:

$$(U'_{\rho})_0=-\left(\frac{fM\rho}{\sqrt{\rho^2-c^2}}\right)_0<0. \tag{3.22}$$

The inequality (3.22) is justified for any values of ρ_0; consequently, circular orbits of arbitrary radius exist within the equatorial plane of the earth.

Let us consider the stability conditions in the form of (3.20). By differentiating we find that

$$\left[\frac{\partial}{\partial\rho}(\rho^3U'_{\rho})\right]_0=-\frac{fM\rho_0^3(\rho_0^2-4c^2)}{\sqrt{(\rho_0^2-c^2)^5}},$$

and therefore condition (3.20) reduces to the following:

$$\frac{2fM\rho_0^3}{(\rho_0^2-c^2)^{3/2}} > \frac{fM\rho_0^3(\rho_0^2-4c^2)}{(\rho_0^2-c^2)^{5/2}} > 0. \tag{3.23}$$

If $\rho_0^2 - 4c^2 > 0$, then the first inequality will be fulfilled. We find that it is justified for

$$\rho_0 > 2c \approx 420 \quad \text{km}, \tag{3.24}$$

i.e., for all types of real satellite motion. Following simple transformations, /302 the left-hand inequality assumes the following form:

$$\rho_0^2 + 2c^2 > 0 \tag{3.25}$$

and, obviously, is always satisfied.

From (3.24) and (3.25) we deduce the existence of Lyapunov stability (with respect to the quantities ρ, $\dot{\rho}$, σ, z, $\dot{z}$) for all circular orbits lying within the equatorial plane of the earth.

It is natural, at this point, to pose the question of whether the stability of circular equatorial orbits of artificial earth satellites may not be disrupted by perturbing forces resulting from the action of terms not taken into account in the potential (3.21) which are dependent upon zonal harmonics of higher order. Such perturbations are defined by the following perturbation function:

$$R = \mu\Phi(\rho, z, \mu), \tag{3.26}$$

where μ is the small parameter on the order of 10^{-6}, and Φ is a bounded, holomorphic function of its variables.

Since, upon addition of the perturbation function (3.26), the differential equations of motion, as before, will possess a kinetic energy integral and also an angular-momentum integral, then, in correspondence with the results of § 2, it can be stated that the circular artificial earth satellite orbits under consideration will be stable in the presence of constantly active perturbations of the type of (3.26), given sufficiently small absolute values of μ.

In particular, stability is not disrupted by terms resulting from the asymmetry of the earth with respect to the equatorial plane. The theorems of § 2 do not enable us to extend the results obtained to perturbations resulting from tesseral or sectorial harmonics -- i.e., to perturbations from the

longitudinal terms. This question will be approached from a different angle in one of the succeeding sections.

Let us consider the circular orbits of the limiting variant of the problem of two immobile centers, in which one of the gravitating centers is infinitely distant (see § 8, Chapter 3). As was already pointed out, the orbits of this problem may be made use of in the capacity of intermediate orbits in several situations: in studying the perturbing influence of the sun, in studying the motion of an artificial space object within a central Newtonian field of forces with constant vector of jet acceleration, and also in studying the effects of light pressure from solar radiation on the motion of artificial earth satellites. We shall give here the results obtained by A. L. Kunitsyn [18]. /303

In a cylindrical coordinate system whose pole is within the gravitating mass, and whose z-axis is directed along the axis of symmetry of the field, the potential will assume the following form:

$$U = \frac{fm}{\sqrt{\rho^2+z^2}} + \frac{fM}{c^2} z. \tag{3.27}$$

The circular orbits

$$\rho = \rho_0, \; z = z_0, \; \sigma = \sigma_0 \tag{3.28}$$

exist provided that

$$\left(\frac{\partial U}{\partial z}\right)_0 = 0, \tag{3.29}$$

from which we obtain the following value for z_0:

$$z_0 = \frac{M\rho_0^3}{mc^2}, \tag{3.30}$$

where c is the constant of areas.

In polar coordinates ($r = \sqrt{\rho^2 + z^2}$ and $\theta = \arcsin \frac{\rho}{r}$) in place of (3.30) we will have

$$r_0^2 = \frac{mc^2}{M} \cos \vartheta_0. \tag{3.31}$$

The equation which we have obtained here represents the envelope of a family of circular orbits, and defines the relationship between the radius of the circular orbit and the displacement of its plane with respect to the center of gravity. From the condition of non-negativity of its right-hand member, we determine the boundaries of the region of existence of such circular motions:

$$-\frac{\pi}{2}<\vartheta_0<\frac{\pi}{2}.$$

Thus, the circular motions will exist only within the region z > 0, so that

$$0<z_0<c\sqrt{\frac{m}{M}}. \tag{3.32}$$

As follows from (3.31), the radius ρ_0 assumes a maximal value when /304

$$\vartheta_{\max}=\arccos\frac{1}{\sqrt{3}}. \tag{3.33}$$

Turning our attention to equation (3.5), we determine the velocity v_0 along the circular orbit. After simple transformations, we obtain the following:

$$\rho_0^3 v_0^2=fm\left[r_0-2\left(\frac{M}{mc^2}\right)^4 r_0^5+\left(\frac{M}{mc^2}\right)^4 r_0^9\right],$$

where $v_0=\rho_0\dot{\phi}_0$. From this we find

$$v_0^2=\frac{fm}{r_0}\left[1-\frac{M}{m}\left(\frac{r_0}{c}\right)^4+\ldots\right].$$

Let us investigation the stability of the motions which have been found. It is obvious that in this case $z_0\neq 0$, and that the necessary and sufficient conditions (3.11) can be transformed to the following:

$$\left[\left(U''_{\rho\rho}+\frac{3}{\rho}U'_\rho\right)U''_{zz}-(U''_{\rho z})^2\right]_0<0,\quad (U''_{zz})_0<0. \tag{3.34}$$

Taking Laplace's equation into consideration, we obtain the following in place of (3.34):

$$\left(\frac{2}{\rho}U'_\rho\right)_0<(U''_{zz})_0<0. \tag{3.35}$$

This condition will be fulfilled if z_0 satisfies the inequality

$$0<z_0<\frac{\sqrt[4]{3}\,c}{9\sqrt{M/m}}. \tag{3.36}$$

In this situation, the circular motions will be stable with respect to ρ, z, $\dot{\rho}$, $\dot{z}$, σ.

From (3.33) and (3.36) we find that circular orbits for which

$$\frac{c\sqrt[4]{3}}{9\sqrt{M/m}} < z_0 < \frac{c}{\sqrt{M/m}}, \tag{3.37}$$

are stable.

§ 4. The Stability of Spheroidal and Hyperboloidal Orbits

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Artificial earth satellites which have small eccentricity are very important in the practical sense. However, the study of these orbits with the usual methods of the classicial theory of perturbations involves certain difficulties. The most general results with the use of these methods in the study of near-circular orbits are those obtained by Strablo [189] and G. V. Samoylovich [190, 191]. Actually, the problem becomes a good deal easier if we disregard the classical theory of perturbations, and instead make use of the solutions of the generalized problem of two immobile centers. In the present section we study the stability of ellipsoidal orbits, which, in the case of small values of the eccentricity of the ellipsoid of rotation, correspond to near-circular orbits [89].

According to § 6, Chapter 3, the differential equations of motion of artificial earth satellites in a normal gravitational field have the following form:

$$\left.\begin{aligned} \frac{d^2v}{d\tau^2} &= \frac{fM}{c^3}\operatorname{ch} v - h\operatorname{sh} 2v - \frac{c_1^2 \operatorname{sh} v}{\operatorname{ch}^3 v}, \\ \frac{d^2u}{d\tau^2} &= h\sin 2u + \frac{c_1^2 \cos u}{\sin^3 u}, \end{aligned}\right\} \tag{4.1}$$

where h and c_1 are arbitrary constants of integration, and the independent variable τ (regularized time) is defined by this differential equation:

$$\frac{d\tau}{dt} = \frac{1}{\operatorname{sh}^2 v + \cos^2 u}. \tag{4.2}$$

Equations of motion similar to (4.1) are quite convenient in the study of the stability of certain types of artificial earth satellite orbits, since the system breaks down into two independent equations. As is evident from equation (4.2), the regularizing variable τ increases monotonically and indefinitely along with t; for this reason, a transfer to regularized time in the present stability problem is possible. Therefore, solutions of equations (4.1) are indefinitely extended, and the variable τ is able to play just the same role in the study of stability as is played by time t.

The arbitrary constants h and c_1 appear in system (4.1). If, in the study of stability, these constants are assumed to be invariable, then we may be able to establish only conditional stability, since the initial and the successive perturbations must satisfy the condition of non-invariability of the total mechanical energy and the condition of constancy of the sector velocity. In order to demonstrate unconditional stability, we make use of the first theorem of § 2. In our proof we shall not supplement system (4.1) with additional equations for h and c_1. A more detailed study of the problem may be found in the present writer's article already referred to [89], and also in the work by V. G. Degtyarev [186]. /306

Equations (4.1) have partial solutions of the form

$$v = v_0, \quad v' = 0, \quad h = h_0, \quad c_1 = c_{10}, \tag{4.3}$$

if v_0 is a root of the equation

$$\frac{fM}{c^3} \operatorname{ch} v_0 - h_0 \operatorname{sh} 2v_0 - \frac{c_{10}^3 \operatorname{sh} v_0}{\operatorname{ch}^3 v_0} = 0. \tag{4.4}$$

Corresponding to solutions (4.3) are orbits which lie upon a compressed spheroid whose axis of symmetry coincides with the earth's axis of rotation. The major semi-axis of this spheroid is c ch v_0, the minor semi-axis is c sh v_0, and the eccentricity is 1/ch v_0. If $c_{10} \neq 0$, then the trajectory "unrolls" onto the ellipsoid, remaining within a certain band which is symmetrical with respect to the equator. Such orbits may be either periodic, in which case they close following a large number of revolutions, or they may be conditionally periodic, in which case they everywhere densely fill the band referred to. The width of this band is determined by the magnitude of the root of the second of the equations of (4.1). This root determines the hyperboloid of revolution, which intersects with the ellipsoid along two symmetrical small circles, which represent the boundary of the region of possible motion. If u is close to $\pi/2$, then the orbit will be located in the vicinity of the equatorial plane of the earth.

If $c_{10} = 0$, the orbits of the artificial earth satellite will lie within meridional planes, and will be in the form of ellipses, whose apses will lie in the equatorial plane of the earth (see § 4, Chapter 4).

Introducing the following designations for perturbations,

$$v - v_0 = x_1, \quad v' = x_2, \tag{4.5}$$

we write the differential equations of perturbed motion as follows: /307

$$\left.\begin{aligned}\frac{dx_1}{dt} &= x_2,\\ \frac{dx_2}{dt} &= -\frac{fM}{c^3}\operatorname{ch}(v_0+x_1) - h_0 \operatorname{sh} 2(v_0+x_1) - \frac{c_{10}^2 \operatorname{sh}(v_0+x_1)}{\operatorname{ch}^3(v_0+x_1)}.\end{aligned}\right\} \tag{4.6}$$

The equations of first approximation have the following form:

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = \left(\frac{4c_{10}^2}{\operatorname{ch}^4 v_0} - \frac{fM \operatorname{ch}^2 v_0}{c^3 \operatorname{sh} v_0}\right) x_1. \tag{4.7}$$

The roots of the characteristic equation of system (4.7) are written as follows:

$$x_{1,2} = \pm \sqrt{\frac{4c_{10}^2}{\operatorname{ch}^4 v_0} - \frac{fM \operatorname{ch}^2 v_0}{c^3 \operatorname{sh} v_0}}. \tag{4.8}$$

If

$$\frac{4c_{10}^2}{\operatorname{ch}^4 v_0} - \frac{fM \operatorname{ch}^2 v_0}{c^3 \operatorname{sh} v_0} \leqslant 0, \tag{4.9}$$

then the ellipsoidal orbits will be stable as regards the first approximation.

The sufficient conditions for stability in this case are found with the help of the first integral of system (4.6):

$$\begin{aligned}V_1 = x_2^2 - \frac{2fM}{c^3}[\operatorname{sh}(v_0+x_1) - \operatorname{sh} v_0] + 2h_0[\operatorname{sh}^2(v_0+x_1) - \\ - \operatorname{sh}^2 v_0] - c_{10}^2\left[\frac{1}{\operatorname{ch}^2(v_0+x_1)} - \frac{1}{\operatorname{ch}^2 v_0}\right].\end{aligned} \tag{4.10}$$

We expand the function V_1 in Maclaurin's series:

$$V_1 = x_2^2 - \left(\frac{4c_{10}^2}{\operatorname{ch}^4 v_0} - \frac{fM \operatorname{ch}^2 v_0}{c^3 \operatorname{sh} v_0}\right) x_1^2 + \ldots \tag{4.11}$$

The condition of its positive definiteness is the inequality (4.9). Consequently, condition (4.9) will be necessary and sufficient for the stability of orbits of the class under consideration.

Condition (4.9), obviously, is not satisfied for meridional (polar) orbits when $c_{10} = 0$. By using the smallness of c, it is possible to demonstrate, however, that this condition is satisfied for all real orbits. The details of the analysis in this case may be found by the reader in [186].

On the basis of the results obtained in § 2, it is possible to demonstrate /308 that the orbits under consideration will be stable in the presence of constantly acting perturbations of the form

$$R = \mu F\left(\frac{\alpha r^2 + \beta z^2 + \gamma}{r^2 + z^2}\right), \tag{4.12}$$

where F is a bounded holomorphic function, α, β, γ are arbitrary constant magnitudes, and μ is a sufficiently small parameter.

From what has been demonstrated it is clear that the ellipsoidal orbits will be stable with respect to the major semi-axis and with respect to the eccentricity of the ellipsoid on the surface of which the artificial earth satellite is moving. However, according to formulas (7.9) and (7.10), Chapter 3, to a constant value of v, with accuracy up to the order of c, there corresponds a constant value of the radius-vector of the artificial earth satellite: this means that the orbits are stable, approximately, with respect to r and r'.

Let us proceed now to a study of hyperboloidal orbits.

The second of the equations of (4.1) has partial solutions of the type

$$u = u_0,\ h = h_0,\ c_1 = c_{10}. \tag{4.13}$$

From the transformation formulas (6.39), Chapter 3, it follows that with $u = u_0$ a moving material point throughout the time of its motion will be found on the surface of the hyperboloid of rotation:

$$\frac{r^2}{c^2 \sin^2 u_0} - \frac{z^2}{c^2 \cos^2 u_0} = 1. \tag{4.14}$$

This particular type of motion we shall refer to as "hyperboloidal". It is easy to see that with $c_{10} = 0$ the motion will take place on the surface of a hyperbola lying within some meridional plane (hyperbolic motion). Properly speaking, neither hyperboloidal nor hyperbolic motion can be described as satellite motion, since during the process of the motion a material particle is removed to infinity. Nevertheless, the orbits of this class under consideration present considerable interests in the theory of motion of interplanetary ships which remain under the influence of the earth.

Solutions of the type of (4.13) exist if u_0 is a root of the

following equation[1]:

$$\sin^4 u_0 = -\frac{c_{10}^2}{2h_0}, \tag{4.15}$$

in other words, with $h_0 > 0$ (in the case in which the total mechanical energy is positive). /309

The equations of perturbed motion will be written as follows:

$$\left.\begin{aligned} \frac{dz_1}{d\tau} &= z_2, \\ \frac{dz_2}{d\tau} &= (h_0 + z_3)\sin 2(u_0 + z_1) + \frac{(c_{10}^2 + z_4)\cos(u_0 + z_1)}{\sin^3(u_0 + z_1)}, \end{aligned}\right\} \tag{4.16}$$

where $u - u_0 = z_1$, $u' = z_2$, $h - h_0 = z_3$, $c_1^2 - c_{10}^2 = z_4$.

System (4.16) has an integral

$$V_1 = z_2^2 + (h_0 + z_3)\cos 2(u_0 + z_1) + \frac{c_{10}^2 + z_4}{\sin^2(u_0 + z_1)}. \tag{4.17}$$

In proving the Lyapunov stability of hyperboloidal orbits, we employ a combination of integral (4.17) and the integrals $V_2 = z_3$, $V_3 = z_4$ in the capacity of a Lyapunov function:

$$V = \frac{4c_{10}^2\cos^2 u_0}{\sin^4 u_0} z_1^2 + z_2^2 - 2\sin 2u_0 \cdot z_1 z_3 - \frac{2\cos u_0}{\sin^3 u_0} z_1 z_4 + \ldots + B_1 z_3^2 + B_2 z_4^2 + \ldots \tag{4.18}$$

For the sine-limitation of this function with sufficiently small values of z_i, the following inequality must be satisfied[2]:

$$\frac{4c_{10}^2\cos^2 u_0}{\sin^4 u_0} > 0. \tag{4.19}$$

It is obvious that this inequality is fulfilled for any hyperboloidal orbits whatever. Consequently, the orbits under consideration will be stable in the Lyapunov sense with respect to the quantities

[1] We leave out of consideration the case $u_0 = 0$, to which corresponds rectilinear motion along the earth's acis of rotation, and also the case $u_0 = \pi/2$, to which corresponds rectilinear motion on the equatorial plane.

[2] It is assumed that B_i are chosen in suitable fashion.

u, u', h and c_1. This implies, in particular, the stability of these orbits with respect to the semi-axes of the hyperboloid on whose surface the motion takes place, and also with respect to the eccentricity of that hyperboloid.

§ 5. The Stability of the Orbits of Artificial Celestial Bodies with Respect to Canonical Elements in the Presence of Constantly Acting Perturbations

In order to describe the perturbed (in the sense of celestial mechanics) motion of a space vehicle conceived as a material point, we make use of one of the systems of canonical elements. For this purpose either the canonical elements of Delaunay, or the first or second system of elements of Poincaré [191] will be found to be convenient. In this study we shall make use of the first system of Poincaré: /310

$$\left.\begin{aligned} L &= \sqrt{f(m_0 + m)a}, \quad \rho_1 = \sqrt{f(m_0 + m)a}\,(1 - \sqrt{1 - e^2}), \\ \rho_2 &= \sqrt{f(m_0 + m)a(1 - e^2)}\,(1 - \cos i), \\ \lambda &= nt + \varepsilon, \quad \omega_1 = -\pi, \quad \omega_2 = -\Omega. \end{aligned}\right\} \tag{5.1}$$

In the formulas of (5.1) we make use of the following designations: m_0 and m are the masses of the mutually gravitating points, a is the major semi-axis of the osculating elliptic orbit, e is the eccentricity of the orbit, i is the inclination, n is the mean motion, Ω is the longitude of the ascending node, π is the longitude of the pericenter, and ε is the mean longitude of the epoch. Taking into consideration the smallness of the mass of the space vehicle m in comparison with the mass of the attracting center, the quantity m in formulas (5.1) can be neglected.

On the basis of [191] we have the following differential equations of perturbed motion:

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial H}{\partial \lambda}, \qquad \frac{d\rho_i}{dt} = \frac{\partial H}{\partial \omega_i}, \\ \frac{d\lambda}{dt} &= -\frac{\partial H}{\partial L}, \quad \frac{d\omega_i}{dt} = -\frac{\partial H}{\partial \rho_i} \qquad (i = 1, 2), \end{aligned}\right\} \tag{5.2}$$

where the Hamiltonian H is defined as follows:

$$H = -\frac{f^2(m_0 + m)^2}{2L^2} + \mu R(L, \rho_1, \rho_2, \lambda, \omega_1, \omega_2). \tag{5.3}$$

The second term in the right-hand member of (5.3) represents the perturbation (perturbing) function, in which the symbol μ denotes the small parameter. If we substitute $\mu = 0$, equations (5.2) will define the unperturbed elliptic orbit.

On account of the conservativeness of the perturbing forces, equations (5.2) have a generalized energy integral:

$$H = \text{const.} \tag{5.4}$$

If the perturbing function satisfies the condition /311

$$\frac{\partial R}{\partial \lambda} = \frac{\partial R}{\partial \omega_1} + \frac{\partial R}{\partial \omega_2}, \tag{5.5}$$

then equations (5.2) possess still another integral:

$$L - \rho_1 - \rho_2 = \text{const.} \tag{5.6}$$

The condition (5.5) means that the perturbing force function is symmetrical with respect to a certain axis, and that the integral of (5.6) represents the angular-momentum integral as expressed in Poincaré elements [191].

We should note that the differential equations (5.2) are useful in describing motions in any limited problem. Their use is indicated in the study of stability in a number of situations: the problem of the motion of an artificial satellite of a spheroidal planet; the three-dimensional circular three-body problem (the case of the motion of a passively gravitating body in the vicinity of one of the gravitating masses); the problem of the motion of a satellite within the gravitational field of a slowly rotating rigid body, whose central ellipsoid of inertia is close to a sphere; and so on.

In studying the system of equations (5.2), we shall make no specific assumptions whatever regarding the physical nature of the perturbing forces. We shall impose only a single limitation on the perturbation function μR, on the assumption that within a certain vicinity of the unperturbed values of the Poincaré elements it is a bounded function.

For $\mu = 0$, the system of equations (5.2) defines a certain unperturbed Keplerian motion. Let us consider the problem of the stability of this motion with respect to a certain number of variables in the presence of constantly acting perturbing forces, as assigned with the help of the perturbation function μR.

First let us consider the case in which the equations (5.2) have only a single integral (5.4). We shall show that in this case the motion will be stable for the assigned perturbing forces with respect to the element L, provided the parameter μ is sufficiently small in absolute value.

Let us supplement the system (5.2) with the equation

$$\dot{\mu} = 0. \tag{5.7}$$

Expanding the system (5.2) and (5.7) we have a partial solution /312

$$\left.\begin{array}{l} L = L_0,\ \rho_1 = \rho_{10},\ \rho_2 = \rho_{20},\ \lambda = \lambda_0(t), \\ \omega_1 = \omega_{10},\ \omega_2 = \omega_{20},\ \mu = 0, \end{array}\right\} \tag{5.8}$$

whose stability with respect to the quantities L and μ must be investigated. In order to do this we sign the following designations for the perturbations of the variables:

$$\left.\begin{array}{l} x_1 = L - L_0,\ x_2 = \mu,\ x_3 = \rho_1 - \rho_{10}, \\ x_4 = \rho_2 - \rho_{20},\ x_5 = \lambda - \lambda_0(t),\ x_6 = \omega_1 - \omega_{10},\ x_7 = \omega_2 - \omega_{20}. \end{array}\right\} \tag{5.9}$$

Then, for the differential equations of perturbed motion, which here we shall not write out in new variables, the following two integrals may be noted:

$$\left.\begin{array}{l} V_1 = \dfrac{f^2(m+m_0)^2}{2}\left[\dfrac{1}{(L_0+x_1)^2} - \dfrac{1}{L_0^2}\right] + x_2 R, \\ V_2 = x_2. \end{array}\right\} \tag{5.10}$$

In order to demonstrate the stability, we construct the Lyapunov function V in the form of a connective of integrals (5.10)

$$V = V_1^2 + AV_2^2, \tag{5.11}$$

where A is an arbitrary constant quantity. Expanding the right-hand portion of formula (5.11) in power series of the perturbations x_1 and x_2, we obtain

$$V = \frac{f^4(m_0+m)^2}{L_0^6}x_1^2 - \frac{2f^2(m_0+m)}{L_0^3}R_0x_1x_2 + (R_0^2+A)x_2^2 + \ldots \tag{5.12}$$

Since the function R, by assumption, is bounded, then it will always be possible to select a numerical value for the constant A such that the quadratic form in the expansion (5.12) will be positive definite. Actually, the condition of sine-restriction of this form is described as follows:

$$R_0^2 + A > 0, \quad \frac{f^4(m_0+m)^2}{L_0^6} > 0. \tag{5.13}$$

The latter inequalities are fulfilled provided A is chosen on the condition /313

$$A > \sup R_0^2.$$

Here, function (5.11) will also be sine-limited with respect to the quantities x_1 and x_2. From this, by virtue of the last theorem of § 2, we conclude that the motion in question is stable with respect to the element L in the presence of constantly acting perturbations. However, from (5.1) it is evident that the motion will also be stable with respect to the major semi-axis of the orbit. But this in turn means that the motion in question is stable in the Legrangian sense[1].

Let us consider a second case. We shall assume that the perturbing function is such that the condition (5.5) is satisfied. Then, in addition to the energy integral, there will exist also an angular-momentum integral (5.6), which in terms of the variables x_i is described as follows:

$$V_3 = x_1 - x_3 - x_4. \tag{5.14}$$

Constructing the Lyapunov function in the form of the connective of the integrals (5.10) and (5.14)

$$V = V_1^2 + A_1V_1^2 + A_2V_2^2. \tag{5.15}$$

where A_1 and A_2 are constants which have been chosen on the basis of considerations analogous to the preceding, we observe that the unperturbed solution is stable in the presence of constantly acting perturbations which satisfy condition (5.5) with respect to L and $(\rho_1 + \rho_2)$. As is evident from formulas (5.1), the unperturbed motion under consideration is stable with respect to the quantities a and $\sqrt{1 - e^2 \cos i}$. From this result, in particular, it follows that if $\rho_2 = 0$ $(i = 0)$ the trajectory is inclosed within a circular annulus whose boundaries are only slightly deformed provided the perturbations of type (5.5) are sufficiently small in absolute value. The results given here are studied in [192].

Note. The integration of differential equations of type (5.2) is carried out in celestial mechanics by one of the methods of successive approximation, and is usually complicated on account of the small divisors, which lead to the appearance of secular terms in the solution. The question naturally arises of whether these secular terms are the result of the physical nature of the problem, or whether they are the result, in certain cases, of a defect in the mathematical methods being used. Of particular interest in this connection is the character of variation in the measure semi-axis with respect to time. This is important, for

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[1] This particular result can also be arrived at directly, without having recourse to the theorems from the theory of stability, simply by analyzing the first integrals. But such a conclusion is justified in any case in which the study of stability is being conducted with the help of integral connectives.

example, in determining the lifetime of the satellite. If the quantity a remains limited during the entire course of the motion, then we can be sure that the motion is stable in the Legrangian sense[1]. From the stability established with respect to the major semi-axis a, it follows that the secular terms in the theory of motion of an artificial earth satellite within a central gravitational field are actually the result of inadequacies in the methods employed.

§ 6. Stability of the Orbits of the Center of Mass of a Sagittal Satellite

Let us consider the problem of the translational-rotating motion of a material homogeneous rectilinear segment of mass m and length $2l$ within the gravitational field of a sphere of mass m_0 which has a spherical distribution of densities. The differential equations of motion in this problem, when stated in terms of a rectangular coordinate system whose origin coincides with the center of mass of the sphere and whose axes are fixed directions, have been obtained by G. N. Duboshin [193]. In cylindrical coordinates these equations are written as follows:

$$\left.\begin{aligned}
\ddot{\rho} - \rho\dot{\lambda}^2 &= -\frac{f(m_0+m)}{r^3}\rho + \frac{m_0+m}{m_0} l^2 \frac{\partial U}{\partial \rho}, \\
\frac{d}{dt}(\rho^2\dot{\lambda}) &= \frac{(m_0+m)}{m_0} l^2 \frac{\partial U}{\partial \lambda}, \\
\ddot{z} &= -\frac{f(m_0+m)}{r^3} z + \frac{m_0+m}{m_0} l^2 \frac{\partial U}{\partial z}, \\
\ddot{\psi}\sin^2\vartheta + \dot{\psi}\dot{\vartheta}\sin 2\vartheta &= 3\frac{\partial U}{\partial \psi}, \\
\ddot{\vartheta} - \dot{\psi}^2 \sin\vartheta\cos\vartheta &= 3\frac{\partial U}{\partial \vartheta},
\end{aligned}\right\} \qquad (6.1)$$

where U is defined as follows:

$$U = \frac{fm_0}{r^3}\sum_{k=1}^{\infty}\frac{P_{2k}(\nu)}{2k+1}\left(\frac{l}{r}\right)^{2k-2}, \qquad (6.2)$$

while /315

$$\nu = \frac{1}{r}[\rho\sin\vartheta\sin(\psi-\lambda) + z\cos\vartheta]. \qquad (6.3)$$

In formulas (6.1) - (6.3) we make use of the following designations: ρ, λ, z are the cylindrical coordinates of the center of inertia of the segment; ψ, ϑ

[1] K. V. Kholshevnikov [207, 208] has established the condition for Lagrangian stability by using the Laplace formula. See also [209].

are the Eulerian angles which define the orientation of the segment within the chosen system of coordinates; f is the constant of gravitation; $P_{2k}(\nu)$ is a Legendre polynomial of the 2k-th order,

$$r = \sqrt{\rho^2 + z^2}$$

The system (6.1) has four first integrals, of which we shall subsequently make use of the energy integral

$$\rho^2 + \rho^2\dot{\lambda}^2 + \dot{z}^2 + \frac{m_0 + m}{3m_0} l^2(\dot{\vartheta}^2 + \dot{\psi}^2 \sin^2 \vartheta) - \frac{2f(m_0 + m)}{r} - \frac{2(m_0 + m)}{m_0} l^2 U = h \qquad (6.4)$$

and one of the integrals of the moment of momentum:

$$\rho^2\dot{\lambda} + \frac{(m_0 + m) l^2}{m_0} \psi \sin^2 \vartheta = c. \qquad (6.5)$$

We shall formulate the stability problem as follows. If $l = 0$, the system (6.1) breaks down into two independent subsystems:

$$\left.\begin{aligned} \ddot{\rho} - \rho\dot{\lambda}^2 &= -\frac{f(m_0 + m)}{r^3}\rho, \\ \frac{d}{dt}(\rho^2\dot{\lambda}) &= 0, \\ \ddot{z} &= -\frac{f(m_0 + m)}{r^3} z \end{aligned}\right\} \qquad (6.6)$$

and

$$\left.\begin{aligned} \ddot{\psi} \sin^2 \vartheta + \dot{\psi}\dot{\vartheta} \sin 2\vartheta &= \frac{fm_0}{r^4}\rho \sin \vartheta \cos(\psi - \lambda) \cdot \nu, \\ \ddot{\vartheta} - \dot{\psi}^2 \sin \vartheta \cos \vartheta &= \\ &= \frac{fm_0}{r^4}[\rho \cos \vartheta \sin(\psi - \lambda) - z \sin \vartheta] \nu. \end{aligned}\right\} \qquad (6.7)$$

The system (6.6) defines the motion of the center of inertia of the segment, when the latter is concentrated at a point. Equations (6.7) do not have any physical meaning, since the rotating motion of a point does not exist. They are nevertheless necessary for mathematical reasons. The question arises whether the orbits described by equations (6.6) and (6.7) are stable with respect to the same variables which characterize the motion of the center of mass of the segment, when l is different from zero, but sufficiently small. The solution of the system (6.1) does not coincide with the solutions of

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equations (6.6) and (6.7), and we therefore consider the question of the stability of the solutions of equations (6.6) and (6.7) in the presence of specific constantly acting perturbations resulting from the shape of the satellite.

In order to arrive at a solution we resort to the method given in § 2 of the present chapter -- in other words we supplement the system (6.1) with the equation

$$\dot{l} = 0 \tag{6.8}$$

and then investigate the stability of the solutions of the enlarged system thus obtained:

$$\left.\begin{gathered}\rho = \rho_0,\ \dot{\rho} = 0,\ \lambda = n(t - t_0),\\ \dot{\lambda} = n = [f(m_0 + m)\rho_0^{-3}]^{1/2},\ z = 0,\ \dot{z} = 0,\\ \psi = \psi_0(l),\ \dot{\psi} = \dot{\psi}_0(l),\ \vartheta = \frac{\pi}{2},\ \dot{\vartheta} = 0,\ l = 0,\end{gathered}\right\} \tag{6.9}$$

where $\psi_0(t)$ represents any partial solution of equation (6.7). The relationships of (6.9) define circular orbits lying within the basic plane of the coordinate system. Since variation in the spatial orientation of the coordinate system does not lead to any change in the form of equations (6.1), or in the form of the force function (6.2), our reasoning applies to all circular orbits of the center of mass of the satellite.

We shall first of all consider the system (6.7) and point out that $\psi_0(t)$ is limited for all initial conditions. The first of the equations of (6.7) yields

$$\ddot{\psi}_0 = \frac{fm_0}{\rho_0^3}\sin(\psi - \lambda)\cos(\psi - \lambda). \tag{6.10}$$

With the help of (6.10) we find that

$$(\dot{\psi} - \dot{\lambda})^2 = h^* + \frac{n^2}{2}\sin^2(\psi - \lambda). \tag{6.11}$$

From (6.11) it is evident that /317

$$n^2 \leqslant (\dot{\psi} - \dot{\lambda})^2 \leqslant h^* + \frac{n^2}{2}.$$

From this it follows that $\dot{\psi}(t)$ is limited.

We now introduce the following designations for the perturbations of the variable quantities:

$$\left.\begin{aligned} &x_1 = \rho - \rho_0, \; x_2 = \rho, && x_3 = \lambda - n(t - t_0), \\ &x_4 = \dot{\lambda} - n, \; x_5 = z, && x_6 = \dot{z}, \qquad x_7 = \vartheta - \frac{\pi}{2}, \\ &x_8 = \dot{\vartheta}, && x_9 = \psi - \psi_0, \; x_{10} = \dot{\psi} - \psi_0, \; x_{11} = l. \end{aligned}\right\} \tag{6.12}$$

Then the system of differential equations of perturbed motion (which we omit for the sake of brevity) has the following first integrals:

$$\begin{aligned} F_1 = {} & (\rho_0 + x_1)^2 (n + x_4)^2 - \rho_0^2 n^2 + x_2^2 + x_6^2 + \\ & + \frac{m_0 + m}{3m_0} x_{11}^2 [x_8^2 + (\psi_0 + x_{10})^2 \cos^2 x_7] - \\ & - \frac{2f(m_0 + m)}{\sqrt{(\rho_0 + x_1)^2 + x_5^2}} - \frac{2(m_0 + m)}{m_0} x_{11}^2 U + \frac{2f(m_0 + m)}{\rho_0}, \end{aligned} \tag{6.13}$$

$$F_2 = (\rho_0 + x_1)^2 (n + x_4) - \rho_0^2 n + \frac{m_0 + m}{m_0} x_{11}^2 (\dot{\psi}_0 + x_{10}) \cos^2 x_7, \tag{6.14}$$

$$F_3 = x_{11}. \tag{6.15}$$

Now let us consider the stability of motion with respect to a certain number of variables, specificially x_1, x_2, x_4, x_5, x_6 and x_{11}. In order to study the stability we shall construct the Lyapunov function V in the form of the connective of the first integrals (6.13) - (6.15)

$$V = F_1 - 2nF_2 + PF_2^2 + MF_3^2, \tag{6.16}$$

whre M and P are arbitrary positive constants which will be selected subsequently. The function V represents the integral of the equations of perturbed motion, and therefore its derivative $\dot{V}$ in virtue of the differential equations is identically equal to zero. If the function V is sine-limited with respect to the chosen variables, the stability of the solutions in question is thereby demonstrated.

Expanding the function V in power series of x_i, we obtain the following: /31

$$V = n^2 (4P\rho_0^2 - 3) x_1^2 + 4Pn\rho_0^3 x_1 x_4 + (P\rho_0^2 + 1)\rho_0^2 x_4^2 + 2n^2 x_5^2 + x_6^2 + N x_{11}^2 + \ldots, \tag{6.17}$$

in which the following designation has been introduced

$$N = M + \frac{m_0 + m}{3m_0} (\dot{\vartheta}_0^2 + \dot{\psi}_0^2) - \frac{2}{3} n^2 P_2(\nu_0) - 2n\dot{\psi}_0 \frac{m_0 + m}{m_0} \tag{6.18}$$

For sufficiently small absolute values of the quantities x_i, the sine of the function V will be determined by the sign of the quadratic form appearing in the expansion (6.17). But this form will be definite positive with respect to the quantities x_1, x_2, x_4, x_5, x_6 and x_{11} if the following inequalities are satisfied:

$$N > \varepsilon > 0, \quad 4P\rho_0^2 > 3 + \varepsilon, \quad P\rho_0^2 > 3 + \varepsilon. \tag{6.19}$$

It has already been shown that $\dot{\psi}_0(t)$ is a limited function of time, and that therefore by an appropriate choice of M the quantity N can be made to be greater than a certain positive number for all values of t. If we choose $P > \frac{3 + \varepsilon}{2\rho_0}$, then the remaining inequalities of (6.19) will also be satisfied.

Thus we have constructed a positive definite function which satisfies the theorem of V. V. Rumyantsev. From this follows the stability of circular orbits of the center of mass of a segment within a Newtonian field of forces with respect to r, $|\dot{r}|$, z, $|\dot{z}|$ and with respect to the angular velocity λ in the presence of constantly acting perturbations resulting from the shape of the satellite, on the assumption that the length of the segment is sufficiently small. This means that when the dimensions of the body are small, rotating motion in the given case will not have any perceptible effect on the motion of the center of inertia.

Consequently, the results of qualitative investigations -- in particular, studies of the stability of motion conducted on the assumption that the satellite is a material point -- are fairly reliable. Here we have limited ourselves to the case of a sagittal satellite moving in a circular orbit; the results might easily be extended to a satellite with sufficiently small dimensions and with arbitrary geometry of masses moving in an elliptical unperturbed orbit. For this purpose we should make use of the Routh theorem and /319 the theorem of canonical transformations of Routhian equations, given in § 4, Chapter 1. It can then be shown that the motion of the satellite will be stable with respect to the elements a, e, i, if its dimensions are sufficiently small.

A study of the stability of translational-rotating motion is given in the works of G. N. Duboshin [193] and V. T. Kondurar' [194-196]. The stability of the rotating motion of a body within a Newtonian field of forces has been studied by a number of authors (see, for example, V. V. Beletskiy [126], V. V. Rumyantsev [197], and V. A. Sarychev [198-200]).

§ 7. Equations of Perturbed Motion for Quasi-Stackel Systems in "Action-Angle" Variables

Let us assume that there exists a dynamic Stackel[1] system, defined by the formulas (5.15) - (5.17) of Chapter 1, in which we should specify

$$\Phi = 0,\ a_i = 1,\ b_i = 0\ (i = 1, 2, \ldots, n). \tag{7.1}$$

Then, according to the second theorem of § 5, Chapter 1, the general solution of the problem is given as follows (see (5.18), Chapter 1).

$$\sum_{i=1}^{n} \int \frac{\varphi_{i1}(q_i)\,dq_i}{\sqrt{\Phi_i(q_i)}} = t + \beta_1, \tag{7.2}$$

$$\sum_{i=1}^{n} \int \frac{\varphi_{ij}(q_i)\,dq_i}{\sqrt{\Phi_i(q_i)}} = \beta_j \qquad (j = 2, \ldots, n), \tag{7.3}$$

where

$$\Phi_i(q_i) = 2U_i(q_i) + \sum_{j=1}^{n} 2\alpha_j \varphi_{ij}(q_i), \tag{7.4}$$

and α_i and β_i are the constants of integration.

Differentiating (7.2) and (7.3) with respect to t, we arrive at

$$\sum_{i=1}^{n} \frac{\varphi_{i1}(q_i)\,dq_i}{\sqrt{\Phi_i(q_i)}} = dt, \tag{7.5}$$

$$\sum_{i=1}^{n} \frac{\varphi_{ij}(q_i)\,dq_i}{\sqrt{\Phi_i(q_i)}} = 0 \qquad (j = 2, 3, \ldots, n). \tag{7.6}$$

We shall assume, in addition, that the case of libration occurs (§ 10, Chapter 1), characterized by simple real roots $q_i = a_i$ and $q_i = b_i$ $(a_i < b_i)$ of the equations /320

$$\Phi_i(q_i) = (q_i - a_i)(b_i - q_i)\psi_i(q_i) = 0, \tag{7.7}$$

in which $\psi_i(a_i) \neq 0$, $\psi_i(b_i) \neq 0$.

We stipulate

$$dw_i = [(q_i - a_i)(b_i - q_i)]^{-\frac{1}{2}} \tag{7.8}$$

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[1] The theory given here of conditional-periodic motions for Stackel systems is discussed in a monograph by Charlier [28].

and introduce the following notation

$$F_{ij}(q_i) = \frac{\varphi_{ij}}{\sqrt{\psi_i(q_i)}}. \tag{7.9}$$

Then equations (7.5) and (7.6) can be rewritten as follows:

$$\sum_{i=1}^{n} F_{i1}\, dw_i = dt, \tag{7.10}$$

$$\sum_{i=1}^{n} F_{ij}\, dw_i = 0 \quad (j = 2, \ldots, n). \tag{7.11}$$

The determinant of this system E by reason of (7.9) is equal to

$$E = \det |F_{ij}| = \frac{1}{\sqrt{\psi_1 \psi_2 \ldots \psi_n}}, \tag{7.12}$$

and, consequently, system (7.10) - (7.11) is solvable with respect to dw_i.

From (7.8) we obtain the following by integrating:

$$q_i = a_i \cos^2 \frac{w_i}{2} + b_i \sin^2 \frac{w_i}{2}. \tag{7.13}$$

Designating

$$G_{ij}(q_i) = \int F_{ij}(q_i)\, dw_i \tag{7.14}$$

and then integrating the system (7.10) - (7.11), we have the following:

$$\sum_{i=1}^{n} G_{i1}(w_i) = t + A_1, \tag{7.15}$$

$$\sum_{i=1}^{n} G_{ij}(w_i) = A_j. \tag{7.16}$$

If in these equations w_k is increased by 2π, then, from (7.15) and (7.16), we can arrive at /321

$$\sum_{i=1}^{n} G_{i1}(w_i) + G_{k1}(w_k + 2\pi) - G_{k1}(w_k) = t + A_1 + 2\omega_{k1}, \tag{7.17}$$

$$\sum_{i=1}^{n} G_{ij}(w_i) + G_{kj}(w_k + 2\pi) - G_{kj}(w_k) = A_j + 2\omega_{jk}, \tag{7.18}$$

where, obviously,

$$\omega_{jk} = \int_{a_k}^{b_k} \frac{\varphi_{kj}(q_k)\,dq_k}{\sqrt{(q_k - a_k)(b_k - q_k)}\,\psi_k(q_k)}, \tag{7.19}$$

for which $\det |\omega_{jk}| \neq 0$.

From (7.13), (7.17) - (7.19) it follows that the coordinates will be conditional-periodic functions of the quantities $t + A_1$, $A_2, \ldots, A_n$, the elementary periods of which are given by the formulas of (7.10) (see § 9, Chapter 1).

We now introduce the new variables u_i with the help of the following relations:

$$\sum_{i=1}^{n} \omega_{i1} u_i = \pi(t + A_1), \tag{7.20}$$

$$\sum_{i=1}^{n} \omega_{ij} u_i = \pi A_j \quad (j = 2, \ldots, n). \tag{7.21}$$

Since the determinant $|\omega_{ij}|$ is different from zero, the latter system can be solved for the variables u_i, and then the generalized coordinates q_i can be represented in the form of functions of u_i:

$$q_i = g_i(u_1, u_2, \ldots, u_n). \tag{7.22}$$

By reason of (7.20) and (7.21) the coordinates q_i will be periodic functions of u_i with the period 2π:

$$g_i(u + 2m_1\pi, u_2 + 2m_2\pi, \ldots, u_n + 2m_n\pi) \equiv g_i(u_1, u_2, \ldots, u_n), \tag{7.23}$$

where m_i are arbitrary whole numbers.

Let us proceed to the compilation of the equations of perturbed motion on the assumption that for such motion the Hamiltonian has the following form: /322

$$H = T - U - \mu R. \tag{7.24}$$

Here the term $T - U$ satisfies the condition of the Stackel theorem (formulas (7.15) - (7.17) Chapter 1 and formulas (7.1) of the present chapter). The motion defined by the Hamiltonian $H_0 = T - U$ will be taken as the unperturbed (intermediate) motion. The function μR will be perturbational (or perturbed),

where μ is the small parameter.

The perturbed motion can be described with differential equations (11.8) Chapter 1, obtained by the method of variation of arbitrary constants (or with the help of the corresponding contact transformation):

$$\dot{\alpha}_i = \mu \frac{\partial R}{\partial \beta_i}, \quad \dot{\beta}_i = -\mu \frac{\partial R}{\partial \alpha_i}. \tag{7.25}$$

However, in qualitative investigations, it is more convenient to use the differential equations for the osculating elements, the role of which is played by the canonical "action-angle" variables. Variables of "angle" type in this case will be the quantities $\eta_i = u_i$, defined by the system (7.20) - (7.21). The canonical variables of action which are conjoint with them will be designated with the symbol ξ_i. For these we will have

$$\xi_i = \frac{1}{2\pi} \int_{q_i}^{v_i} \sqrt{2\varphi_i(q_i)}\, dq_i. \tag{7.26}$$

The canonical character of this transformation can easily be established with the help of Jacobi theorem (see § 4, Chapter 1) in the form given by Poincaré [191]. If the canonical transformation from variables α_i, β_i to the variables ξ_i, η_i is given by the derivative of the function $S(t, \alpha_1, \ldots, \alpha_n, \xi_1, \ldots, \xi_n)$, then its total variation will be equal to:

$$\delta S = \sum_{i=1}^{n} \left(\frac{\partial V}{\partial \alpha_i} \delta\alpha_i + \frac{\partial V}{\partial \xi_i} \delta\xi_i \right). \tag{7.27}$$

Here the left-hand member, with the help of the formulas (4.11) Chapter 1, can be transformed to the following form: /323

$$\sum_{i=1}^{n} (\beta_i \delta\alpha_i - \eta_i \delta\xi_i). \tag{7.28}$$

This brings us to the conclusion that the transformation of the variables will be canonical provided the expression (7.28) represents the total differential of any function. Substituting in expression (7.28) the values of the new variables ξ_i, η_i, as assigned by formulas (7.20), (7.21) and (7.26), we observe that this expression in unperturbed motion will be constant and equal to the total mechanical energy α_1, taken with opposite sign. This means that the derivative function of the contact transformation S will be written as follows:

$$S = -\alpha_1 t, \tag{7.29}$$

where α_1 should be regarded as a function of the new variables.

As a result of this transformation, the equations of perturbed motion can be represented as follows:

$$\frac{d\xi_i}{dt} = \frac{\partial K}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{\partial K}{\partial \xi_i}, \tag{7.30}$$

where the new Hamiltonian is assigned by the formula

$$K = -\alpha_1 + \mu R. \tag{7.31}$$

Assuming the Hamiltonian to be an analytical function of the small parameter μ, it can be expanded in power series of μ:

$$K = K_0 + \mu K_1 + \ldots, \tag{7.32}$$

in which, by reason of (7.22) and (7.23) the functions K_1, K_2,... will be periodic with respect to η_i, with the period 2π, while K_0 will depend only upon ξ_i.

§ 8. The Stability of Satellite Orbits in the Presence of Constantly Acting Conservative Perturbations /324

The reasoning advanced in this section will be based upon the Kolmogrov-Arnol'd theorem, which can be formulated as follows.

Let the motion of a conservative mechanical system be defined by the Hamiltonian

$$H = H_0(p_1, \ldots, p_n) + \mu H_1(p_1, \ldots, p_n, q_1, \ldots, q_n, \mu), \tag{8.1}$$

in which μ is the small parameter, while the function H_1 (p, q, μ) has the period 2π with respect to q_1,..., q_n and is analytical within a certain region G: $|\mathrm{Im}\ q| < \rho$, $p \in P$, and within this region let the following condition of non-degeneracy be fulfilled:

$$\det \left| \frac{\partial^2 H_0}{\partial p_i \partial p_j} \right| \neq 0, \tag{8.2}$$

while

$$|\mu H_1| < M. \tag{8.3}$$

Then, with sufficiently small values of $|M|$, the points of the region G, excluding the set which is of small measure along with M, will move conditional-periodically along n-dimensional tori which are close to the tori

$$p = \text{const.}$$

The theorem formulated here has been proven by V. I. Arnol'd [166, 168], using a modified form of Newton's method of successive approximations. In addition, Arnol'd was able to prove a more general theorem in which no requirement of non-degeneracy (8.2) is imposed on the system of Hamiltonian equations. This general theorem is equally applicable to non-degenerate systems and to systems with limiting or eigen degeneracy (see § 10, Chapter 1). In the great majority of cases in celestial mechanics and celestial ballistics, the Kolmogrov-Arnol'd theorem is inapplicable since here we are concerned with limiting or eigen degeneracy.

The latter difficulty can be avoided if, instead of Keplerian unperturbed orbits, we make use of other types of intermediate orbits for which the condition (8.2) will be fulfilled. Examples of such orbits are found in the case of the classical or the generalized problem of two immobile centers, and of its limiting variants. Here, for the sake of simplicity, we shall proceed on the basis of the limiting variant of the generalized problem of two immobile centers, which coincides with the Barrar problem (see § 4, Chapter 3). /325

We shall study the motion of a satellite considered as a material point in a coordinate system with fixed directions of axes; as the z-axis we shall take the axis of rotation of the planet, while the origin of coordinates will be placed at one of the spherical points of inertia, which any planet will possess if we ignore its equatorial compression.

In the chosen system of coordinates the gravitational potential of the planet is expressed by the following formula (see § 4, Chapter 3):

$$U = \frac{fM}{r} + \frac{fMz_c \sin\varphi}{r^2} + \mu R(r, \varphi, \lambda, \mu). \tag{8.4}$$

Here r, ϕ, λ, are the spherical coordinates in the system of reference, μ is the small parameter, and μR is the perturbation function. We should note that in the remainder of the investigation the actual nature of the perturbing forces is not an important factor. Therefore, the function R may include within itself both the perturbing action of the shape of the planet (which is not allowed for by the first two terms of the potential (8.4)), and the various other perturbing factors.

Taking as our generalized coordinates the quantities

$$q_1 = r, \quad q_2 = \varphi, \quad q_3 = \sin\lambda, \tag{8.5}$$

for the total integral of the Hamiliton+Jacobi equation in the unperturbed problem under consideration -- the problem is determined by the first two terms of the potential (8.4) -- according to (7.12) Chapter 3 we will have the following:

$$V = -\alpha_1 t + \sqrt{2\alpha_3}\arcsin q_3 + \\ + \sqrt{2}\int \sqrt{\alpha_1 q_1^2 + fmq_1 + \alpha_2}\frac{dq_1}{q_1} + \\ + \sqrt{2}\int \sqrt{(\alpha_2 + fmz_c \sin q_2)\cos^2 q_2 - \alpha_3}\frac{dq_2}{\cos q_2}, \tag{8.6}$$

where α_1, α_2, α_3 are the constants of integration.

We shall now transform the equations of motion, which are not repeated here, to the form of (7.30). To do this we introduce the canonical "action-angle" variables in the unperturbed problem at hand. Since here we have a partial case of a Stackel system, then the motion of the satellite will be conditional-periodic, while the "action-angle" variables will be expressed in terms of the elementary periods which are defined by formula (7.19): /326

$$\left.\begin{gathered}
\omega_{11} = \frac{1}{\sqrt{2}}\int_{a_1}^{b_1}\frac{q_1 dq_1}{\sqrt{P_1(q_1)}}, \quad \omega_{12} = -\frac{1}{\sqrt{2}}\int_{a_1}^{b_1}\frac{dq_1}{q_1\sqrt{P_1(q_1)}}, \\
\omega_{13} = 0, \\
\omega_{21} = 0, \quad \omega_{22} = \frac{1}{\sqrt{2}}\int_{a_2}^{b_2}\frac{\cos q_2 dq_2}{\sqrt{P_2(q_2)}}, \\
\omega_{23} = -\frac{1}{\sqrt{2}}\int_{a_2}^{b_2}\frac{dq_2}{\cos q_2\sqrt{P_2(q_2)}}, \\
\omega_{31} = 0, \quad \omega_{32} = 0, \quad \omega_{33} = \frac{1}{\sqrt{2\alpha_3}}\int_{-1}^{1}\frac{dq_3}{\sqrt{1-q_3^2}}.
\end{gathered}\right\} \tag{8.7}$$

In formulas (8.7) a_1 and b_1 are the roots of the equation

$$P_1(q_1) = \alpha_1 q_1^2 + fmq_1 - \alpha_2 = 0, \tag{8.8}$$

between which, during the process of motion, is included the coordinate q_1, while a_2 and b_2 are the analogous roots of the equation

$$P_2(q_2) = \cos^2 q_2\,(fmz_c \sin q_2 + \alpha_2) - \alpha_3 = 0. \tag{8.9}$$

As follows from (7.20) and (7.21), the canonical variables η_i are associated with the elementary periods ω_{ij} as follows:

$$\left.\begin{aligned}\eta_1 &= \frac{\pi}{\omega_{11}}(\beta_1 - t),\\ \eta_2 &= \frac{\pi}{\omega_{22}}\left[\beta_2 - \frac{\omega_{12}}{\omega_{11}}(\beta_1 - t)\right],\\ \eta_3 &= \frac{\pi}{\omega_{33}}\left\{\beta_3 - \frac{\omega_{23}}{\omega_{22}}\left[\beta_2 - \frac{\omega_{12}}{\omega_{11}}(\beta_1 - t)\right]\right\},\end{aligned}\right\} \tag{8.10}$$

where the symbol β_i denotes the canonical elements which are conjugate with α_i (see formula (4.19) Chapter 1). The variables of action ξ_i, in correspondence with (7.26) in the given problem will be expressed in terms of α_i as follows: /327

$$\xi_i = \frac{1}{\pi}\int_{a_i}^{b_i}\sqrt{2P_i}\,dq_i \quad (i = 1, 2), \quad \xi_3 = \frac{1}{\pi}\int_0^{\pi}\sqrt{2\alpha_3}\,d\lambda. \tag{8.11}$$

If we find α_i from equation (8.11) as a function of ξ_i, then the contact transformation which interests us will be written as a derivative function:

$$S = -\alpha_1(\xi_1, \xi_2, \xi_3)t, \tag{8.12}$$

and the canonical equations of perturbed motion will assume the form of (7.30), while the new characteristic function K will be defined as follows:

$$K = \mu R + \frac{\partial S}{\partial t} = \mu R - \alpha_1(\xi_1, \xi_2, \xi_3). \tag{8.13}$$

In these variables, the equations of unperturbed (in the sense indicated above) motion will assume the following form:

$$\frac{d\xi_i}{dt} = 0, \quad \frac{d\eta_i}{dt} = -\frac{\partial\alpha_1}{\partial\xi_i} = \omega_i(\xi_1, \xi_2, \xi_3). \tag{8.14}$$

From (8.10) and (8.13) it is evident that

$$\omega_1 = \frac{\pi}{\omega_{11}}, \quad \omega_2 = \frac{\pi\omega_{12}}{\omega_{11}\omega_{22}}, \quad \omega_3 = -\frac{\pi\omega_{12}\omega_{23}}{\omega_{11}\omega_{22}\omega_{33}}. \tag{8.15}$$

To assure ourselves of the applicability of the Kolmogorov-Arnol'd theorem, it is necessary to demonstrate that the Hess determinant (8.2) is identically not equal to zero. As follows from (8.2) and (8.14), the problem reduces to calculating the values of the Jacobian

$$\det\left|\frac{\partial\omega_i}{\partial\xi_j}\right|. \tag{8.16}$$

This calculation can be considerably simplified if the canonical variables ξ_i are expressed in terms of the osculating elements[1] a, p, i, which are defined by the formulas (7.17), (7.18) and (7.20) Chapter 3. We now have /328

$$\alpha_1 = -\frac{fm}{2a}, \quad \alpha_2 = \frac{fmp}{2}, \quad \alpha_3^2 = \frac{fm}{2}(p + 2z_c \sin i)\cos^2 i. \tag{8.17}$$

Since this transformation is non-singular,

$$\frac{D(a, p, i)}{D(\xi_1, \xi_2, \xi_3)} \neq 0, \tag{8.18}$$

while the Jacobian (8.16) is represented as follows:

$$\det\left|\frac{\partial\omega_i}{\partial\xi_j}\right| = \frac{D(\omega_1, \omega_2, \omega_3)}{D(a, p, i)} \cdot \frac{D(a, p, i)}{D(\xi_1, \xi_2, \xi_3)}. \tag{8.19}$$

Then the condition of non-degeneracy is written as follows:

$$\frac{D(\omega_1, \omega_2, \omega_3)}{D(a, p, i)} \neq 0. \tag{8.20}$$

With the help of formulas (8.7) and (8.17), it is not difficult to obtain the following:

$$\left.\begin{aligned} \omega_{11} &= \frac{\pi}{n}, \quad \omega_{12} = \frac{\pi}{\sqrt{fmp}}, \quad \omega_{22} = \sqrt{\frac{2}{fmz_c(\lambda_1 - \lambda_3)}}\,\mathbb{K}(k), \\ \omega_{23} &= \frac{1}{\sqrt{2fmz_c(\lambda_1 - \lambda_3)}}\left\{\frac{1}{1-\lambda_1}\Pi(n', k) + \frac{1}{1+\lambda_1}\Pi(n'', k),\right. \end{aligned}\right\} \tag{8.21}$$

where K(k) is a total elliptical integral of the first type, $\Pi(n', k)$, $\Pi(n'', k)$ are total elliptic integrals of the third type, while for the module and the parameters of these integrals we obtain the following:

$$k^2 = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_3}, \quad n' = \frac{\lambda_1 - \lambda_2}{1 - \lambda_1}, \quad n'' = \frac{\lambda_2 - \lambda_1}{1 + \lambda_1}. \tag{8.22}$$

In formulas (8.22) the symbols λ_1, λ_2, λ_3 denote the roots of the equation

[1] These osculating elements are certainly not Keplerian, since the unperturbed orbits in this case are orbits of the limiting variant of the generalized problem of two immobile centers, and not the trajectories of the two-body problem.

(8.9) as transformed with the help of (8.17) to the following form:

$$2z_c\lambda^3 + p\lambda^2 - 2z_c\lambda - (p \sin i - 2z_c \cos^2 i) \sin i = 0. \tag{8.23}$$

From (8.15) and (8.21), it is evident that ω_1 does not depend upon p and i, and therefore the condition (8.20) can be represented as follows:

$$\frac{D(\omega_1, \omega_2, \omega_3)}{D(a, p, i)} = \frac{\partial\omega_1}{\partial a} \cdot \frac{D(\omega_2, \omega_3)}{D(p, i)} \neq 0, \tag{8.24}$$

or, taking into consideration the fact that $\omega_1 = \pi/n$, while $\partial\omega_1/\partial a \neq 0$, /329

$$\frac{D(\omega_2, \omega_3)}{D(p, i)} \neq 0. \tag{8.25}$$

Since in the case of close artificial satellites the ratio z_c/p is on the order of $3 \cdot 10^{-3}$ (see § 6, Chapter 3) and diminishes with increase in the major semi-axis of the orbit, the frequencies ω_2 and ω_3 can be expanded in Taylor series in powers of the quantity z_c/p:

$$\left.\begin{aligned} \omega_2 &= n\left[1 + \left(\frac{z_c}{p}\right)^2\left(1 - \frac{7}{4}\sin^2 i\right)\right] + \ldots, \\ \omega_3 &= -n\left(1 - \frac{z_c}{p}\sin i\right) + \ldots \end{aligned}\right\} \tag{8.26}$$

Then the condition of non-degeneracy (8.25) is written as follows:

$$\frac{D(\omega_2, \omega_3)}{D(p, i)} = \frac{2n^2 \cos i}{p}\left(\frac{z_c}{p}\right)^3 + \ldots \tag{8.27}$$

This condition will be fulfilled when $i \neq \pi/2$.

Then, in correspondence with the Kolmogorov-Arnol'd theorem, for a sufficiently small absolute value of the perturbing function μR we can assert that the perturbed motion of the satellite will be conditional-periodic and stable with respect to the variables of action ξ_i for the majority of the initial conditions. The perturbed motion of the satellite of a planet will take place along tori which are close to unperturbed tori:

$$\xi_i = \text{const}. \tag{8.28}$$

This assertion is justified only for those orbits for which the frequencies

ω_i do not satisfy any relation of the form

$$\sum_{i=1}^{3} k_i \omega_i = 0; \tag{8.29}$$

where k_i are whole coefficients which are not all equal to zero at the same time.

In the problem at hand the Hess determinant, when using canonical variables of "action-angle" type is not equal to zero; nevertheless, for certain initial conditions, the frequencies may satisfy (8.29) -- that is, the possibility of random degeneracy is not excluded. This may happen, for example in the case of initial conditions which correspond to periodic motions for which the frequencies ω_i are proportional. For satellite orbits with such initial conditions, the Kolmogorov-Arnol'd theorem is not applicable, and a more detailed investigation is necessary. /330

However, if we limit ourselves to a narrower class of perturbing forces, the Arnol'd method can be used, and a better result can be obtained. For example, in the potential (8.4) let us assume that the perturbation function μR does not depend upon the longitude λ: i.e., both in the unperturbed and in the perturbed problems the force field is axisymmetrical. This is the situation in the case of the problem of the motion of a satellite of a spheroidal planet. The coordinate λ here is cyclic, and therefore it can be ignored. Following simple transformations, we arrive at the following altered Hamiltonian:

$$H = \frac{1}{2}\left(p_1^2 + \frac{1}{q_1^2} p_2^2\right) - \frac{\alpha_3^2}{2q_1^2 \cos^2 q_2} - \frac{fmz_c \sin q_2}{q_1^2} - \mu R(q_1, q_2, \mu), \tag{8.30}$$

in which α_3 is the constant of the cyclic integral (the angular-momentum integral).[3] As a result of reducing the order of the system of equations of motion, we have the following:

$$\frac{d\xi_i}{dt} = \frac{\partial K}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = -\frac{\partial K}{\partial \xi_i} \quad (i = 1, 2), \tag{8.31}$$

where

$$K = \mu R + \frac{\partial S}{\partial t}, \tag{8.32}$$

where the symbol S denotes the derivative function of the canonical transformation. We should note that for all transformations the constant α_3 remains the same. The variables η_i, as above, are determined by formulas (8.10) while ξ_i are determined by formulas (8.11).

Equations (8.31) have a form which is necessary for the application of the Kolmogorov-Arnol'd theorem. Substituting $\mu = 0$ in (8.31) we obtain the following equations of motion: /331

$$\frac{d\xi_i}{dt} = 0, \quad \frac{d\eta_i}{dt} = \omega_i \quad (i = 1, 2), \tag{8.33}$$

where the frequencies ω_i are determined by formulas (8.15). These equations in phase space ξ_i, η_i, define motion along the two-dimensional torus $\xi_i = \text{const}$ $(i = 1, 2)$.

Here the condition of non-degeneracy (8.2) can be represented as follows:

$$\frac{D(\omega_1, \omega_2)}{D(\xi_1, \xi_2)} \neq 0. \tag{8.34}$$

Since ω_{22} is independent of α_2, and ω_{11} is independent of α_1, in place of (8.24) we obtain

$$\frac{\partial \omega_{11}}{\partial \alpha_1} \frac{\partial}{\partial \alpha_2}\left(\frac{\omega_{22}}{\omega_{12}}\right) \neq 0. \tag{8.35}$$

As before, it is possible to demonstrate that this condition is fulfilled for all initial conditions. From this we conclude that the motion under consideration will be stable in the sense of Arnol'd provided the cyclic constant α_3 is not perturbed.

However, the latter limitation can readily be removed. In order to do this, we represent the constant α_3 in the form of a power series with respect to the small parameter (up to the point at which the cyclic coordinate can be ignored)

$$\alpha_3 = \alpha_3^{(0)} + \mu\alpha_3^{(1)} + \ldots, \tag{8.36}$$

having taken $\alpha_3^{(0)}$ for the unperturbed value of the area constant. The remaining terms of the series are included in the perturbation function. All other considerations remain unchanged.

Following V. I. Arnol'd, we can assert that in the given case the three-dimensional invariant manifold $H = h$, where h is fixed, unfolds along two-dimensional tori $\xi_1 = \text{const}$ into narrow toroidal layers, while the trajectory does not leave that layer in which it was found at the initial moment. Consequently, the motion will be stable in the Lagrangian sense. This result is

justified only for perturbations defined by a sufficiently small (in absolute value) perturbation function of the type $\mu R(r, \phi, \mu)$.

Note. The Kolmogorov-Arnol'd theorem in a very definite sense supplements the Poincaré theorem on the nonexistence of single-valued analytical integrals of Hamiltonian systems, which differ from classical first integrals (or those which follow from them [201]). According to the Poincaré theorem, if the Hess determinant of the canonical equations of type (7.30) - (7.31) is not equal to zero, then the latter, generally speaking do not possess any other single-valued analytical integral, apart from the Jacobi integral (or one equivalent to it). /332

For certain supplemental limitations imposed on the Hamiltonian function, it is possible to assert that the problem of the motion of a satellite of an axisymmetrical "earth" is without any first single-valued integrals which are analytic with respect to the small parameter μ, apart from the kinetic-energy integral and the angular-momentum integral, provided that μ is sufficiently small.

If we take the equations of the Barrar problem and ignore the cyclic constant, then the equations of motion will have the form (8.31). These equations admit only of a first integral, namely the kinetic-energy integral, and the Hess determinant of this system will be different from zero. Therefore, the basic condition of the Poincaré theorem is fulfilled.

The supplemental limitation referred to above is that in any arbitrarily small portion of the region there exist an infinite set of ratios k_1/k_2, where k_1 and k_2 are whole numbers for which not all the coefficients $\int_0^{2\pi} R\, d\eta_i$ become zero when the latter become secular. We limit ourselves here to remarks, without aducing the corresponding proof.

One point which deserves attention is the fact that conclusions drawn from the Poincaré theorem, both in celestial mechanics and in celestial ballistics, are primarily of formal mathematical interest, since the Poincaré theorem is of a local character. The degree of smallness of the parameter μ is quite important. This is evident from the following example. In the problem of the motion of an artificial earth satellite, we pointed out two integrable cases (see Chapter 3): the Barrar case, and the generalized problem of two immobile centers. Although these two cases are related in the mechanical sense, on the formal mathematical plane they are entirely independent and separate problems. Both of these problems are integrated in closed form, in quadratures. If we take the Barrar problem as the simplified problem, the equations of perturbed motion for canonical variables of "angle-action" type, for a sufficiently small value of μ, should not admit of any new integrals, apart from the energy integral and the angular-momentum integral. Despite this fact, the generalized problem of two immobile centers is integrable, even though its potential differs from the Barrar potential by a quantity on the order of 10^{-6}. Consequently, in the given case the parameter on the order of /333

10^{-6}, cannot be considered small. Therefore, for practical purposes the question of the existence of first integrals which are different from the classical integrals, should, be considered an open one as far as celestial mechanics and celestial ballistics are concerned.

The research carried out by V. I. Arnol'd in this area opens up definite possibilities in two directions. First, we are able to distinguish the qualitative properties of motion. Second, the modified method of Newton which is used by V. I. Arnol'd, can also be employed in order to arrive at solutions, keeping in mind the objective of rapid convergence. We should ask ourselves which of these two possibilities inherent in the Arnol'd method is the most fruitful and valuable.

It is true that in any problem described by canonical equations, we can establish stability with respect to a certain group of variables, employing the assistance of Arnol'd's results; but this does not mean that in certain particular problems the investigation of stability based entirely on Arnol'd's methods, is not of interest. In this connection, one advantage of using the Arnol'd method is that it enables us to delineate those distinguishing quantities which remain approximately the same in the case of perturbed motion and in the case of unperturbed motion. If, in the study of one and the same mechanical problem, the unperturbed portion of the Hamiltonian is delineated in two different ways, then it is possible to obtain a series of different combinations of the parameters of motion which do not vary widely in the case of conservative perturbations.

As regards the expedience of using the method developed by V. I. Arnol'd, we can only say that, as far as calculations are concerned, the matter remains open, and that the practical adaptability of the method remains unclear.

What is given above represents a study of the stability of satellite motions within a central gravitational field, on the basis of the Barrar problem. Analogous research might be carried out on the basis of the generalized problem of two immobile centers. The results given in this section have already been published in part by the present author [202]. Analogous properties of the motion of artificial earth satellites can be determined by still another method, employing the theorems of J. Moser [43], as has been done by V. T. Kyner [203]. /334

§ 9. The Stability of Motion in the Limited Circular Three-Body Problem[1]

We have noted the significance of the circular 3 body problem in space ballistics. In this section, we will analyze the Arnold stability of the orbits

[1] The results of this section were reported by the present writer at the conference on celestial mechanics held in Kiev in January 1965.

of this problem. Our main attention will be given to the specifics of the Arnold problem, without going into technical details. This Arnold problem was first solved by A.M. Leontovich [204], who investigated the stability of the triangle of libration points [201]. As in the preceding section, unperturbed motion will be selected such that the Hess determinant of the Hamiltonian equations of unperturbed motion is other than zero. The unperturbed portion of the Hamiltonian can be selected in at least two ways, which allows us to study two types of motion important for the applications.

We shall assume, first of all, that the actively gravitating masses (see § 1, Chapter 1) m_1 and m_2 satisfy the condition $m_1 >> m_2$. This happens, in particular, in the "earth-moon-spaceship" problem, and also in the "sun-Jupiter-asteroid" astronomical problem. We shall concern ourselves only with the case in which the constant of the Jacobi integral (formula (6.3) Chapter 6) h is negative -- i.e., when the energy with respect to the motion of the passively gravitating point is negative. Then, from a consideration of the curve of zero velocity (see (9.2) Chapter 3) it follows that motion is possible either in a limited vicinity of the body of small mass, the case of "earth-moon-artificial moon satellite", or within a region embracing a large mass ("sun-Jupiter-inner planet")[1].

If we consider a satellite with small mass but with sufficiently large /33 mean motion, such as might be used to orbit close to the moon, as the small parameter we can take the ratio of the mean motion of the gravitating masses to the mean motion of the satellite. Then, to the perturbed portion of the Hamiltonian, it is expedient to relate those terms which are proportional to the angular velocity of rotation of the gravitating masses. The Hamiltonian function of the unperturbed problem in a rotating system of coordinates rigidly associated with the gravitating masses (see (6.1) Chapter 6), will coincide with the Hamiltonian of the classical problem of two immobile centers.

In the second type the motion takes place in the vicinity of the greater mass. In this case we consider two bodies in a rotating system of axes as the unperturbed problem. The perturbing portion of the Hamiltonian will contain terms which are proportional to the smaller of the two actively gravitating masses, m_2. The introduction of rotating axes during the analysis of the plane motion, eliminates eigen degeneracy (see § 10, Chapter 1), since as a result the general solution of the unperturbed problem will depend not upon a single "frequency", equal to the mean motion, but upon two separate frequencies (the second frequency will be determined by the angular velocity of rotation of the coordinate system). The equations of motion (6.1), Chapter 6 can be transformed to canonical form:

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[1] We assume also that h has a numerical value corresponding to the delineated types. We shall not consider any other types of motion.

$$\frac{dx}{dt} = \frac{\partial H}{\partial \dot{x}}, \quad \frac{d\dot{x}}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial \dot{y}}, \quad \frac{d\dot{y}}{dt} = -\frac{\partial H}{\partial y}, \tag{9.1}$$

where the Hamiltonian H is equal to

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + n(\dot{x}y - \dot{y}x) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}. \tag{9.2}$$

In (9.1) and (9.2) for the sake of simplicity we shall choose the system of units in such a way that $fm_1 = 1 - \mu$, $fm_2 = \mu$.

Now we shall perform two contact transformations in succession. Transformation from variables x, $\dot{x}$, y, $\dot{y}$ to the new canonical variables ξ_1, ξ_2, η_1, η_2 we shall assign by the derivative function

$$S = \xi_1(\dot{x}\cos\xi_2 + \dot{y}\sin\xi_2). \tag{9.3}$$

The second canonical transformation to the variables q_1, q_2, p_1, p_2 we shall define by the derivative function /336

$$W = p_2\xi_2 + \int_{p_1(p_1 - \sqrt{p_1^2 - p_2^2})}^{\xi_1} \sqrt{-\frac{p_2^2}{\tau} + \frac{2}{\tau} - \frac{1}{p_1^2}}\, d\tau. \tag{9.4}$$

It should be noted that q_1 is the mean anomaly of a point moving along an ellipse which it would describe around a point of the larger mass located at the center of inertia of the gravitating bodies in the case of unaltered initial conditions; q_2 is the longitude of the line of apsides reckoned from the x-axis and $p_1 = \sqrt{a}$, $p_2 = \sqrt{a(1 - e^2)}$, where a and e, respectively, are the major semi-axis and the eccentricity of the indicated ellipse.

In these variables, instead of system (9.1) we will have

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{9.5}$$

Here the Hamiltonian function H is obtained from (9.2) upon making the indicated transformations, and is expanded in power series of μ:

$$H = H_0 + \mu H_1 + \ldots, \tag{9.6}$$

in which all H_i are functions which are periodic with respect to q_i, having the

period 2π. Thus, the variables p_i and q_i which have been introduced will be canonical variables of "action-angle" type.

The first term in the right-hand member of formula (9.6) is determined as follows:

$$H_0 = -\frac{1}{2p_1^2} - np_2 \tag{9.7}$$

on the basis of the assumption made above, it is the Hamiltonian of the unperturbed problem. The perturbation function is

$$R = \sum_{i=1}^{\infty} \mu^i H_i. \tag{9.8}$$

From (9.7) it is evident that the Hess determinant is identically equal to zero, and that, consequently, the Kolmogorov-Arnol'd cannot be applied directly. Therefore, the system (9.5) is first subjected to still another transformation. Specifying that /337

$$V = V(H), \tag{9.9}$$

where V is an arbitrary holomorphic function with respect to H, instead of system (9.5) we arrive at

$$\frac{dq_i}{dt} = \frac{1}{V'(h)}\frac{\partial V}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{1}{V'(h)}\frac{\partial V}{\partial q_i}. \tag{9.10}$$

In this process of transformation we have made use of the Jacobi system (9.5) $H = h$.

On the basis of the assumed holomorphic character of function V, it follows that

$$V = \sum_{k=0}^{\infty} \mu^k V_k. \tag{9.11}$$

It is obvious that

$$V_0 = V\ (H_0(p_1, p_2)). \tag{9.12}$$

Equations (9.10) admit of a generalized energy integral,

$$V/V' = h'. \tag{9.13}$$

We should note that, in general, solutions of the system (9.10) are not solutions of equations (9.5). Of the solutions of system (9.10), only those for which the following relationship holds satisfy the system of differential equations of motion:

$$V(h) - h'V'(h) = 0. \tag{9.14}$$

This, however, does not prevent our investigation of the stability of motion, since, if the solutions of system (9.10) are stable, then the same thing will be true of the solutions of the initial system.

In order to avoid limiting our investigation of stability to a family of isoenergetic trajectories, it is possible to proceed as follows. Let h' be chosen on the basis of the condition (9.14), and let the constant h be represented in the form of a series: /338

$$h = \sum_{k=0}^{\infty} h_k \mu^k. \tag{9.15}$$

Then, considering only those solutions of (9.10) which at the same time are solutions of system (9.5), we shall include terms of the form $h_k\mu^k$ in the perturbed portion of the Hamiltonian. This makes it possible, in the study of solutions, to consider all those which are close to unperturbed solutions as regards magnitude of energy.

For $\mu = 0$, we obtain the following differential equations of unperturbed motion from (9.10):

$$\frac{dq_i}{dt} = \frac{1}{V'(h)}\frac{\partial V_0}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{1}{V'(h)}\frac{\partial V_0}{\partial q_i}. \tag{9.16}$$

Since $\partial V_0/\partial q_i \equiv 0$, from (9.16) we find that

$$p_i = \beta_i = \text{const}. \tag{9.17}$$

Then, from the first group of equations of (9.16), we obtain

$$q_i = \frac{1}{V'(h)}\frac{\partial V_0}{\partial \beta_i} t + \alpha_i. \tag{9.18}$$

From (9.17) - (9.18) it follows that in the phase space p_i, q_i the motion takes place along two-dimensional tori $p_i = \beta_i$, and that it is conditional-periodic.

Making use of the Kolmogorov-Arnol'd theorem, we demonstrate that the motion defined by the Hamiltonian H_0 will also be conditional-periodic, and

that it will take place along tori which are close to the tori $p_i = \beta_i$. In order to do this it is necessary to establish the non-degeneracy of the problem -- i.e., the fulfillment of condition (8.2).

With the help of (9.7) and (9.9).we find that

$$\frac{\partial^2 V_0}{\partial p_1^2} = V_0'' \cdot \frac{1}{p_1^6} - V_0' \cdot \frac{3}{p_1^4}, \quad \frac{\partial^2 V_0}{\partial p_2^2} = n^2 V_0'', \quad \frac{\partial^2 V_0}{\partial p_1 \partial p_2} = -\frac{n}{p_1^3} V_0'',$$

from which

$$\det \left| \frac{\partial^2 V_0}{\partial p_i \partial p_j} \right| = -\frac{3n^2}{p_1^4} V_0' V_0''. \tag{9.19}$$

From (9.19) it is evident that it is always possible to select the function $V_0(H)$ in such a way that

$$\det \left| \frac{\partial^2 V_0}{\partial p_i \partial p_j} \right| \neq 0.$$

Thus, the problem under study does not admit of eigen or limiting degeneracy, and therefore the motion will be stable in the sense of Arnol'd with respect to the quantities p_1, p_2. /339

The limitation on the magnitude of the perturbing mass can be removed, if we consider only the class of motions for which the unperturbed orbits embrace both gravitating masses. For this type of motion it is necessary, before hand, to transform the equations in motion (6.1), Chapter 6.

Denoting the barycentric radius-vector of the passively gravitating point with the symbol r, and expanding the potential in series of Legendre polynomials, we obtain

$$U = \frac{1}{r} + \sum_{k=2}^{\infty} \frac{\gamma_k}{r^{k+1}} P_k\left(\frac{x}{r}\right). \tag{9.20}$$

In formula (9.20) the coefficients γ_k are defined as follows:

$$\gamma_k = \mu (1 - \mu) [\mu^{k-1} - (-1)^{k-1} (1 - \mu)^{k-1}]. \tag{9.21}$$

The expansion of (9.21) converges absolutely and uniformly for all

$$r > \max \{\mu; 1 - \mu\}.$$

We now shift from the variables x, y, t to new variables ξ, η, τ with the help of

$$\xi = \varepsilon x,\ \eta = \varepsilon y,\ \tau = \sqrt{\varepsilon^3}t, \tag{9.22}$$

where ε is a parameter. Then, instead of systems (6.1) Chapter 6, we will have

$$\left.\begin{aligned} \frac{d^2\xi}{d\tau^2} - 2\nu\frac{d\eta}{d\tau} - \nu^2\xi + \frac{\xi}{\rho^3} &= \frac{\partial R}{\partial \xi}, \\ \frac{d^2\eta}{d\tau^2} + 2\nu\frac{d\xi}{d\tau} - \nu^2\eta + \frac{\eta}{\rho^3} &= \frac{\partial R}{\partial \eta}, \end{aligned}\right\} \tag{9.23}$$

where the following designations are used:

$$\nu = \frac{n}{\sqrt{\varepsilon^3}}, \quad \rho = \sqrt{\xi^2 + \eta^2}. \tag{9.24}$$

Taking the quantity ε as the small parameter, it is possible to retrace the preceding arguments almost literally, and, in correspondence with the Kolmogorov-Arnol'd theorem, to conclude that in this case, given sufficiently small values of the parameter ε, that the motion will be conditional-periodic and stable in the sense of Arnol'd with respect to the variables of action $\sqrt{a}$, $\sqrt{p}$. Here, no limitations whatever are placed on the magnitudes of the gravitating masses. The requirements of smallness placed upon the parameter ε means that the motion will be stable and conditional-periodic for all those orbits which embrance both gravitating masses, provided the radii of the masses are sufficiently large. /340

Now we shall present a second method for the study of stability (in the sense of Arnol'd) of the orbits of the circular three-body problem. Here, also, no limitations will be placed on the magnitude of the gravitating masses.

Let us consider the three-dimensional variant of the circular three-body problem, the differential equations of which have the following form:

$$\ddot{x} - 2n\dot{y} - n^2x = U'_x,\ \ddot{y} + 2n\dot{x} - n^2y = U'_y, \qquad \ddot{z} = U'_z, \tag{9.25}$$

where the force function, according to (1.25) and (1.26) Chapter 1 may be written as follows:

$$U = \frac{fm_1}{r_1} + \frac{fm_2}{r_2}.$$

The distance between the gravitating masses will be designated with the symbol 2c.

When n = 0, equations (9.25) define motion in the problem of two immobile centers (see § 2, Chapter 3). Since a large number of types of motion exist in this problem, we shall select that particular type in which the passively gravitating point moves in the vicinity of one of the gravitating masses (for example, close to mass m_2). This is the way in which we shall consider the case of satellite motion. If the moving point is sufficiently close to mass m_2, then its mean angular velocity of rotation n_0 will be large in comparison with n, and in this case it will be possible to select the quantity $\varepsilon = n/n_0$ as the small parameter. Equations (9.25) are then transformed to the following form:

$$\left.\begin{aligned} \frac{d^2x}{d\tau^2} - 2\varepsilon\frac{dy}{d\tau} - x &= \frac{1}{n_0^2}U'_x, \\ \frac{d^2y}{d\tau^2} + 2\varepsilon\frac{dx}{d\tau} - y &= \frac{1}{n_0^2}U'_y, \\ \frac{d^2z}{d\tau^2} &= \frac{1}{n_0^2}U'_z, \end{aligned}\right\} \tag{9.26}$$

where $\tau = n_0 t$.

Let us study the simplified system of equations obtained from (9.26) when $\varepsilon = 0$. /34 We shall introduce the spheroidal coordinates u, v, w as defined by formulas (3.29) Chapter 1. In these variables, the kinetic energy and the force function of the problem will be as follows (see (2.5) Chapter 3 and (3.36) Chapter 1):

$$T = \frac{c^2}{2}[(\operatorname{ch}^2 v - \cos^2 u)(\dot{u}^2 + \dot{v}^2) + \operatorname{sh}^2 v \sin^2 u \cdot \dot{w}^2], \tag{9.27}$$

$$U = \frac{1}{c(\operatorname{ch}^2 v - \cos^2 u)}[f(m_1 + m_2)\operatorname{ch} v + f(m_2 - m_1)\cos u]. \tag{9.28}$$

Integrating the problem of two immobile centers is possible with the help of the Stackel theorem. As a result, we arrive at the following form for the total integral of the Hamilton-Jacobi:

$$\begin{aligned} W = &\int \sqrt{2f(m_1+m_2)c\operatorname{ch} v + 2\left(\alpha_1 c^2 \operatorname{ch}^2 v + \alpha_2 + \frac{\alpha_3}{\operatorname{sh}^2 v}\right)}\, dv + \\ &+ \int \sqrt{-2f(m_1-m_2)c\cos u + 2\left(-\alpha_1 c^2 \cos^2 u - \alpha_2 + \frac{\alpha_3}{\sin^2 u}\right)}\, du + \\ &+ \sqrt{-2\alpha_3}\, w, \end{aligned} \tag{9.29}$$

in which α_i represents arbitrary constants.

For elementary periods, in correspondence with formula (7.19) of the present chapter, we have

$$\left.\begin{aligned}
&\omega_{11}=c^2\int_{v_1}^{v_2}\frac{\operatorname{ch}^2 v\,dv}{\sqrt{F_1(v)}},\quad \omega_{12}=\int_{v_1}^{v_2}\frac{dv}{\sqrt{F_1(v)}},\quad \omega_{13}=\int_{v_1}^{v_2}\frac{dv}{\operatorname{sh}^2 v\sqrt{F_1(v)}};\\
&\omega_{21}=-c^2\int_{u_1}^{u_2}\frac{\cos^2 u\,du}{\sqrt{F_2(u)}},\quad \omega_{22}=-\int_{u_1}^{u_2}\frac{du}{\sqrt{F_2(u)}},\quad \omega_{23}=\int_{u_1}^{u_2}\frac{du}{\sin^2 u\sqrt{F_2(u)}},\\
&\omega_{31}=0,\quad \omega_{32}=0,\quad \omega_{33}=\frac{2\pi}{\sqrt{-2\alpha_3}},
\end{aligned}\right\} \tag{9.30}$$

where the symbols v_1 and v_2 denote the roots of the equation

$$F_1(v)=2f(m_1+m_2)c\operatorname{ch}v+2\left(\alpha_1c^2\operatorname{ch}^2 v+\alpha_2+\frac{\alpha_3}{\operatorname{sh}^2 v}\right)=0, \tag{9.31}$$

while the symbols u_1 and u_2 denote the roots of the equation /342

$$F_2(u)=-2fc(m_1+m_2)\cos u+2\left(-\alpha_1c^2\cos^2 u-\alpha_2+\frac{\alpha_3}{\sin^2 u}\right)=0. \tag{9.32}$$

The frequencies corresponding to canonical variables of "angle" type will be as follows:

$$\omega_1=\frac{\pi}{\omega_{11}},\quad \omega_2=\frac{\pi\omega_{12}}{\omega_{11}\omega_{22}},\quad \omega_3=\frac{\pi\omega_{12}\omega_{23}}{\omega_{11}\omega_{22}\omega_{33}}, \tag{9.33}$$

while the variables of action are assigned as follows:

$$\eta_1=\frac{1}{\pi}\int_{v_1}^{v_2}\sqrt{F_1(v)}dv,\quad \eta_2=\frac{1}{\pi}\int_{u_1}^{u_2}\sqrt{F_2(u)}du,\quad \eta_3=2\sqrt{-2\alpha_3}. \tag{9.34}$$

The quantities η_i in the phase space of variables of "action-angle" type, determine those tori on which unperturbed motion takes place.

It is possible to demonstrate that in the problem under consideration the conditions of the Kolmogorov-Arnol'd theorem are fulfilled. In particular, the following Jacobian will be different from zero:

$$\frac{D(\omega_1,\omega_2,\omega_3)}{D(\eta_1,\eta_2,\eta_3)}. \tag{9.35}$$

Consequently, for sufficiently small values of ε unperturbed motion will be conditional-periodic and stable (in the sense of Arnol'd) with respect to the variables η_i. From this we conclude, in particular, that the use of the problem of two immobile centers as the unperturbed aspect in the solution of the circular three-body problem is entirely justified.

Note. Since the differential equations of motion of a satellite of a "triaxial" planet, rotating with constant angular velocity around one of the main central axes of inertia, have the same mathematical structure as the differential equations of the circular three-body problem, being distinguished only by the perturbing terms in the force function, the results obtained in the present section apply with equal force to both problems. Here it should be assumed that the "triaxiality" of the planet is fairly small. The qualitative properties of the orbits of the circular three-body problem, embracing both gravitating masses, will be the same as the corresponding properties of equatorial orbits of artificial satellites.

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